

Quantitative Automata under Probabilistic Semantics

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Abstract

Automata with monitor counters, where the transitions do not depend on counter values, and nested weighted automata are two expressive automata-theoretic frameworks for quantitative properties. For a well-studied and wide class of quantitative functions, we establish that automata with monitor counters and nested weighted automata are equivalent. We study for the first time such quantitative automata under probabilistic semantics. We show that several problems that are undecidable for the classical questions of emptiness and universality become decidable under the probabilistic semantics. We present a complete picture of decidability for such automata, and even an almost-complete picture of computational complexity, for the probabilistic questions we consider.

1 Introduction

Traditional to quantitative verification. While traditional formal verification focused on Boolean properties of systems, such as “every request is eventually granted”, recently significant attention has been shifted to quantitative aspects such as expressing properties like “the long-run average success rate of an operation is at least one half” or “the long-run average (or the maximal, or the accumulated) resource consumption is below a threshold.” Quantitative properties are essential for performance related properties, for resource-constrained systems, such as embedded systems.

Overview. The first natural way to express quantitative properties is to consider automata with counters. However, computational analysis of such models quickly lead to undecidability, and a classical way to limit expressiveness for decidability is to consider *monitor counters*, i.e., the counter values do not influence the control. The second approach is to consider automata with weights (or weighted automata). However, weighted automata have limited expressiveness, and they have been extended as nested weighted automata [18] (nesting of weighted automata) for expressiveness. We establish that for a well-studied and wide class of quantitative functions, automata with monitor counters and nested weighted automata are equivalent, i.e., they represent a robust class of quantitative specifications. We study for the first time such quantitative automata under probabilistic semantics. Quite surprisingly we show that several problems that are undecidable for the classical questions of emptiness and universality become decidable under the probabilistic semantics. We present a complete picture of decidability for nested weighted automata and automata with monitor counters under probabilistic semantics.

Automata with monitor counters. A natural extension of automata is automata with monitor counters, which are automata equipped with counters. At each transition, a counter can be started, terminated, or the value of the counter can be increased or decreased. However, the transitions do not depend on the counter values, and hence they are referred to as monitor counters. The values of the counters when they are terminated gives rise to the sequence of weights. A value function aggregates the sequence into a single value. For example, for words over $\{a, \#\}$, such automata can express the maximal length of block of a ’s that appear infinitely often. Automata with monitor counters are similar in spirit with the class of register automata of [2], and we consider them over infinite words.

Weighted automata. Weighted automata extend finite automata where every transition is assigned a rational number called weight. Hence every run gives rise to a sequence of weights, and which is aggregated into a single value by a value function. For non-deterministic weighted automata, the value of a word w is the infimum value of all runs over w . Weighted automata provide a natural and flexible framework for

expressing quantitative¹ properties [16]. First, weighted automata were studied over finite words with weights from a semiring, and ring multiplication as value function [21], and later extended to infinite words with limit averaging or supremum as value function [16, 15, 13]. While weighted automata over semirings can express several quantitative properties [28], they cannot express long-run average properties that weighted automata with limit averaging can [16]. However, even weighted automata with limit averaging cannot express the following basic quantitative property (the example is from [18]).

Example 1. Consider infinite words over $\{r, g, i\}$, where r represents requests, g represents grants, and i represents idle. A basic and interesting property is the average number of i 's between a request and the corresponding grant, which represents the long-run average response time of the system.

Nested weighted automata. To enrich expressiveness, weighted automata was extended to *nested weighted automata (NWA)* [18]. A nested weighted automaton consists of a master automaton and a set of slave automata. The master automaton runs over input infinite words. At every transition the master can invoke a slave automaton that runs over a finite subword of the infinite word, starting at the position where the slave automaton is invoked. Each slave automaton terminates after a finite number of steps and returns a value to the master automaton. Each slave automaton is equipped with a value function for finite words, and the master automaton aggregates the returned values from slave automata using a value function for infinite words. For Boolean finite automata, nested automata are equivalent to the non-nested counterpart, whereas nested weighted automata is strictly more expressive than non-nested weighted automata [18], for example, nested weighted automata can express the long-run average response time property (see [18, Example 5]). It has been shown in [18] that nested weighted automata provides a specification framework where many basic quantitative properties, that cannot be expressed by weighted automata, can be expressed easily, and it provides a natural framework to study quantitative run-time verification.

Classical questions. The classical questions for automata are *emptiness* (resp., *universality*) that asks for the existence (resp., non-existence) of words that are accepted. Their natural extensions has been studied in the quantitative setting as well (such as for weighted automata, NWA, etc) [16, 18].

Motivation for probabilistic questions. One of the key reasons for quantitative specification is to express performance related properties. While the classical emptiness and universality questions express the best/worst case scenarios (such as the best/worst-case trace of a system for average response time), they cannot express the average case average response time, where the average case represents a probability distribution over the traces. Performance related properties are of prime interest for probabilistic systems, and quite surprisingly, quantitative automata have not been studied in a probabilistic setting which we consider in this work.

Probabilistic questions. Weighted automata and its extension as nested weighted automata, or automata with monitor counters are all measurable functions from infinite words to real numbers. We consider probability distribution over infinite words, and as a finite representation for probability spaces we consider the classical model of finite-state Markov chains. Moreover, Markov chains are a canonical model for probabilistic systems [26, 4]. Given a measurable function (or equivalently a random variable), the classical quantities w.r.t. a probability distribution are: (a) the expected value; and (b) the cumulative distribution below a threshold. We consider the computation of the above quantities when the function is given by a nested weighted automata or automata with monitor counters, and the probability distribution is given by a finite-state Markov chain. We also consider the approximate variants that ask to approximate the above quantities within a tolerance term $\epsilon > 0$. Moreover, for the cumulative distribution we consider the special case of *almost-sure* acceptance, which asks whether the probability is 1.

Our contributions. In this work we consider several classical value functions, namely, SUP, INF, LIMSUP, LIMINF, LIMAVG for infinite words, and MAX, MIN, SUM, SUM^B, SUM⁺ (where SUM^B is the sum bounded by B , and SUM⁺ is the sum of absolute values) for finite words. First, we establish translations (in both directions) between automata with monitor counters and a special class of nested

¹We use the term “quantitative” in a non-probabilistic sense, which assigns a quantitative value to each infinite run of a system, representing long-run average or maximal response time, or power consumption, or the like, rather than taking a probabilistic average over different runs.

weighted automata, where at any point only a bounded number of slave automata can be active. However, in general, in nested weighted automata unbounded number of slave automata can be active. We describe our main results for nested weighted automata.

- *LIMSUP and LIMINF functions.* We consider deterministic nested weighted automata with LIMSUP and LIMINF functions for the master automaton, and show that for all value functions for finite words that we consider, all probabilistic questions can be answered in polynomial time. This is in contrast with the classical questions, where the problems are PSPACE-complete or undecidable (see Remark 16 for further details).
- *INF and SUP functions.* We consider deterministic nested weighted automata with SUP and INF functions for the master automaton, and show the following: the approximation problems for all value functions for finite words that we consider are $\#P$ -hard and can be computed in EXPTIME; other than the SUM function, the expected value, the distribution, and the almost-sure problems are PSPACE-hard and can be solved in EXPTIME; and for the SUM function, the above problems are uncomputable. Again we establish a sharp contrast w.r.t. the classical questions as follows: for the classical questions, the complexity of LIMSUP and SUP functions always coincide, whereas we show a substantial complexity gap for probabilistic questions (see Remark 24 and Remark 25 for further details).
- *LIMAVG function.* We consider deterministic nested weighted automata with LIMAVG function for the master automaton, and show that for all value functions for finite words that we consider, all probabilistic questions can be answered in polynomial time. Again our results are in contrast to the classical questions (see Remark 29).
- *Non-deterministic automata.* For non-deterministic automata we show two results: first we present an example to illustrate the conceptual difficulty of evaluating a non-deterministic (even non-nested) weighted automata w.r.t. a Markov chain, and also show that for nested weighted automata with LIMSUP value function for master automaton and SUM value function for slave automata, all probabilistic questions are undecidable (in contrast to the deterministic case where we present polynomial-time algorithms).

Note that from above all decidability results we establish carry over to automata with monitor counters, and we show that all our undecidability (or uncomputability) results also hold for automata with monitor counters. Note that decidability results for nested weighted automata are more interesting as compared to automata with monitor counters as unbounded number of slaves can be active. Our results are summarized in Table 2 (in Section 6.2), Table 3 (in Section 6.3), and Table 4 (in Section 6.4). In summary, we present a complete picture of decidability of the basic probabilistic questions for nested weighted automata (and automata with monitor counters).

Technical contributions. We call a nested weighted automaton \mathbb{A} , an $(f; g)$ -automaton if its the master automaton value function is f and the value function of all slave automata is g . We present the key details of our main technical contributions, and for sake of simplicity here explain for the case of the uniform distribution over infinite words. Our technical results are more general though (for distribution given by Markov chains).

- We show that in a deterministic (LIMINF; SUM)-automaton \mathbb{A} , whose master automaton is strongly connected as a graph, almost all words have the same value which, is the infimum over values of any slave automaton from \mathbb{A} over all finite words.
- For a deterministic (INF; SUM)-automaton \mathbb{A} and $C > 0$ we define \mathbb{A}^C as the deterministic (INF; SUM)-automaton obtained from \mathbb{A} by stopping every slave automaton if it exceeds C steps. We show that for every deterministic (INF; SUM)-automaton \mathbb{A} and $\epsilon > 0$, there exists C exponential in $|\mathbb{A}|$ and polynomial in ϵ such that the expected values of \mathbb{A} and \mathbb{A}^C differ by at most ϵ .
- We show that the expected value of a deterministic (LIMAVG; SUM)-automaton \mathbb{A} coincides with the expected value of the following deterministic (non-nested) LIMAVG-automaton \mathcal{A} . The automaton \mathcal{A} is obtained from \mathbb{A} by replacing in every transition an invocation of a slave automaton \mathfrak{B} by the weight equal to the expected value of \mathfrak{B} .

Related works. Quantitative automata and logic have been extensively and intensively studied in recent years. The book [21] presents an excellent collection of results of weighted automata on finite words. Weighted automata on infinite words have been studied in [16, 15, 22]. The extension to weighted automata with monitor counters over finite words has been considered (under the name of cost register automata) in [2]. A version of nested weighted automata over finite words has been studied in [7], and nested weighted automata over infinite words has been studied in [18]. Several quantitative logics have also been studied, such as [6, 8, 1]. While a substantial work has been done for quantitative automata and logics, quite surprisingly none of the above works consider the automata (or the logic) under probabilistic semantics that we consider in this work. Probabilistic models (such as Markov decision processes) with quantitative properties (such as limit-average or discounted-sum) have also been extensively studied for single objectives [24, 30], and for multiple objectives and their combinations [20, 11, 17, 9, 19, 10, 25, 12, 3, 5]. However, these works do not consider properties that are expressible by nested weighted automata (such as average response time) or automata with monitor counters.

In the main paper, we present the key ideas and main intuitions of the proofs of selected results, and detailed proofs are relegated to the appendix.

2 Preliminaries

Words. We consider a finite *alphabet* of letters Σ . A *word* over Σ is a (finite or infinite) sequence of letters from Σ . We denote the i -th letter of a word w by $w[i]$. The length of a finite word w is denoted by $|w|$; and the length of an infinite word w is $|w| = \infty$.

Labeled automata. For a set X , an X -labeled automaton \mathcal{A} is a tuple $\langle \Sigma, Q, Q_0, \delta, F, C \rangle$, where (1) Σ is the alphabet, (2) Q is a finite set of states, (3) $Q_0 \subseteq Q$ is the set of initial states, (4) $\delta \subseteq Q \times \Sigma \times Q$ is a transition relation, (5) F is a set of accepting states, and (6) $C : \delta \mapsto X$ is a labeling function. A labeled automaton $\langle \Sigma, Q, q_0, \delta, F, C \rangle$ is *deterministic* if and only if δ is a function from $Q \times \Sigma$ into Q and Q_0 is a singleton. In definitions of deterministic labeled automata we omit curly brackets in the description of Q_0 and write $\langle \Sigma, Q, q_0, \delta, F, C \rangle$.

Semantics of (labeled) automata. A *run* π of a (labeled) automaton \mathcal{A} on a word w is a sequence of states of \mathcal{A} of length $|w| + 1$ such that $\pi[0]$ belong to the initial states of \mathcal{A} and for every $0 \leq i \leq |w| - 1$ we have $(\pi[i], w[i], \pi[i + 1])$ is a transition of \mathcal{A} . A run π on a finite word w is *accepting* iff the last state $\pi[|w|]$ of the run is an accepting state of \mathcal{A} . A run π on an infinite word w is *accepting* iff some accepting state of \mathcal{A} occurs infinitely often in π . For an automaton \mathcal{A} and a word w , we define $\text{Acc}(w)$ as the set of accepting runs on w . Note that for deterministic automata, every word w has at most one accepting run ($|\text{Acc}(w)| \leq 1$).

Weighted automata. A *weighted automaton* is a \mathbb{Z} -labeled automaton, where \mathbb{Z} is the set of integers. The labels are called *weights*. We assume that weights are given in the unary notation, and, hence, the values of weights are linearly bounded in the size of weighted automata.

Semantics of weighted automata. We define the semantics of weighted automata in two steps. First, we define the value of a run. Second, we define the value of a word based on the values of its runs. To define values of runs, we will consider *value functions* f that assign real numbers to sequences of rationals. Given a non-empty word w , every run π of \mathcal{A} on w defines a sequence of weights of successive transitions of \mathcal{A} , i.e., $C(\pi) = (C(\pi[i - 1], w[i], \pi[i]))_{1 \leq i \leq |w|}$; and the value $f(\pi)$ of the run π is defined as $f(C(\pi))$. We denote by $(C(\pi))[i]$ the weight of the i -th transition, i.e., $C(\pi[i - 1], w[i], \pi[i])$. The value of a non-empty word w assigned by the automaton \mathcal{A} , denoted by $\mathcal{L}_{\mathcal{A}}(w)$, is the infimum of the set of values of all *accepting* runs; i.e., $\inf_{\pi \in \text{Acc}(w)} f(\pi)$, and we have the usual semantics that infimum of an empty set is infinite, i.e., the value of a word that has no accepting run is infinite. Every run π on an empty word has length 1 and the sequence $C(\pi)$ is empty, hence we define the value $f(\pi)$ as an external (not a real number) value \perp . Thus, the value of the empty word is either \perp , if the empty word is accepted by \mathcal{A} , or ∞ otherwise. To indicate a particular value function f that defines the semantics, we will call a weighted automaton \mathcal{A} an f -automaton.

Value functions. We will consider the classical functions and their natural variants for value functions. For finite runs we consider the following value functions: for runs of length $n + 1$ we have

1. *Max and min:* $\text{MAX}(\pi) = \max_{i=1}^n (C(\pi))[i]$ and $\text{MIN}(\pi) = \min_{i=1}^n (C(\pi))[i]$.
2. *Sum, absolute sum and bounded sum:* the sum function $\text{SUM}(\pi) = \sum_{i=1}^n (C(\pi))[i]$, the absolute sum $\text{SUM}^+(\pi) = \sum_{i=1}^n \text{Abs}((C(\pi))[i])$, where $\text{Abs}(x)$ is the absolute value of x , and the bounded sum value function returns the sum if all the partial absolute sums are below a bound B , otherwise it returns the exceeded bound $-B$ or B , i.e., formally, $\text{SUM}^B(\pi) = \text{SUM}(\pi)$, if for all prefixes π' of π we have $\text{Abs}(\text{SUM}(\pi')) \leq B$, otherwise $\text{SUM}^B(\pi) = \text{sgn} \cdot B$ where sgn is the sign of the shortest prefix whose sum is outside $[-B, B]$.

We denote the above class of value functions for finite words as $\text{FinVal} = \{\text{MAX}, \text{MIN}, \text{SUM}^B, \text{SUM}\}$.

For infinite runs we consider:

1. *Supremum and Infimum, and Limit supremum and Limit infimum:* $\text{SUP}(\pi) = \sup\{(C(\pi))[i] : i > 0\}$, $\text{INF}(\pi) = \inf\{(C(\pi))[i] : i > 0\}$, $\text{LIMSUP}(\pi) = \limsup\{(C(\pi))[i] : i > 0\}$, and $\text{LIMINF}(\pi) = \liminf\{(C(\pi))[i] : i > 0\}$.
2. *Limit average:* $\text{LIMAVG}(\pi) = \limsup_{k \rightarrow \infty} \frac{1}{k} \cdot \sum_{i=1}^k (C(\pi))[i]$.

We denote the above class of value functions for infinite words as $\text{InfVal} = \{\text{SUP}, \text{INF}, \text{LIMSUP}, \text{LIMINF}, \text{LIMAVG}\}$.

Silent moves. Consider a $(\mathbb{Z} \cup \{\perp\})$ -labeled automaton. We can consider such an automaton as an extension of a weighted automaton in which transitions labeled by \perp are *silent*, i.e., they do not contribute to the value of a run. Formally, for every function $f \in \text{InfVal}$ we define $\text{sil}(f)$ as the value function that applies f on sequences after removing \perp symbols. The significance of silent moves is as follows: it allows to ignore transitions, and thus provide robustness where properties could be specified based on desired events rather than steps.

3 Extensions of weighted automata

In this section we consider two extensions of weighted automata, namely, automata with monitor counters and nested weighted automata.

3.1 Automata with monitor counters

Automata with monitor counters are intuitively extension of weighted automata with counters, where the transitions do not depend on the counter value. We define them formally below.

Automata with monitor counters. An automaton with n monitor counters $\mathcal{A}^{\text{m-c}}$ is a tuple $\langle \Sigma, Q, Q_0, \delta, F \rangle$ where (1) Σ is the alphabet, (2) Q is a finite set of states, (3) $Q_0 \subseteq Q$ is the set of initial states, (4) δ is a finite subset of $Q \times \Sigma \times Q \times (\mathbb{Z} \cup \{s, t\})^n$ called a transition relation, (each component refers to one monitor counter, where letters s, t refer to starting and terminating the counter, respectively, and the value from \mathbb{Z} is the value that is added to the counter), and (5) F is the set of accepting states. Moreover, we assume that for every $(q, a, q', \vec{u}) \in \delta$, at most one component in \vec{u} contains s , i.e., at most one counter is activated at each position. Intuitively, the automaton $\mathcal{A}^{\text{m-c}}$ is equipped with n counters. The transitions of $\mathcal{A}^{\text{m-c}}$ do not depend on the values of counters (hence, we call them monitor counters); and every transition is of the form (q, a, q', \vec{v}) , which means that if $\mathcal{A}^{\text{m-c}}$ is in the state q and the current letter is a , then it can move to the state q' and update counters according to v . Each counter is initially inactive. It is activated by the instruction s , and it changes its value at every step by adding the value between $-N$ and N until termination t . The value of the counter at the time it is terminated is then assigned to the position where it has been activated. An automaton with monitor counters $\mathcal{A}^{\text{m-c}}$ is *deterministic* if and only if Q_0 is a singleton and δ is a function from $Q \times \Sigma$ into $Q \times (\mathbb{Z} \cup \{s, t\})^n$.

Semantics of automata with monitor counters. A sequence π of elements from $Q \times (\mathbb{Z} \times \{\perp\})^n$ is a *run* of $\mathcal{A}^{\text{m-c}}$ on a word w if (1) $\pi[0] = \langle q_0, \vec{\perp} \rangle$ and $q_0 \in Q_0$ and (2) for every $i > 0$, if $\pi[i-1] = \langle q, \vec{u} \rangle$ and $\pi[i] = \langle q', \vec{u}' \rangle$ then $\mathcal{A}^{\text{m-c}}$ has a transition $(q, w[i], q', \vec{v})$ and for every $j \in [1, n]$ we have (a) if $v[j] = s$, then $u[j] = \perp$ and $u'[j] = 0$, (b) if $v[j] = t$, then $u[j] \in \mathbb{Z}$ and $u'[j] = \perp$, and (c) if $v[j] \in \mathbb{Z}$, then $u'[j] = u[j] + v[j]$. A run π is *accepting* if some state from F occurs infinitely often on the

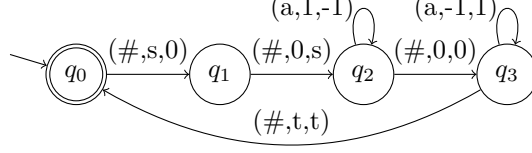


Figure 1: The automaton $\mathcal{A}_{\text{diff}}$ computing the maximal difference between the lengths of blocks of a 's at odd and the following even positions.

first component of π , infinitely often some counter is activated and every activated counter is finally terminated. An accepting run π defines a sequence π^W of integers and \perp as follows: let the counter started at position i be j , and let the value of the counter j terminated at the earliest position after i be x_j , then $\pi^W[i]$ is x_j . The semantics of automata with monitor counters is given, similarly to weighted automata, by applying the value function to π^W .

Remark 2. Automata with monitor counters are very similar in spirit to the register automata considered in the works of [2]. The key difference is that we consider infinite words and value functions associated with them, whereas previous works consider finite words. Another key difference is that in this work we will consider probabilistic semantics, and such semantics has not been considered for register automata before.

Example 3 (Blocks difference). Consider an alphabet $\Sigma = \{a, \#\}$ and a language \mathcal{L} of words $(\#^2 a^* \# a^* \#)^\omega$. On the words from \mathcal{L} we consider a quantitative property “the maximal block-length difference between odd and even positions”, i.e., the value of word $\#^2 a^{n[1]} \# a^{n[2]} \#^3 \dots$ is $\sup_{0 \leq i} |n[2 * i + 1] - n[2 * i + 2]|$. This property can be expressed by a SUP-automaton $\mathcal{A}_{\text{diff}}$ with two monitor counters depicted in Figure 1.

The automaton $\mathcal{A}_{\text{diff}}$ has a single initial state q_0 , which is also the only accepting state. It processes the word w in subwords $\#^2 a^k \# a^m \#$ in the following way. First, it reads $\#^2$ upon which it takes transitions from q_0 to q_1 and from q_1 to q_2 , where it starts counters 1 and 2. Next, it moves to the state q_2 where it counts letters a incrementing counter 1 and decrementing counter 2. Then, upon reading $\#$, it moves to q_3 , where it counts letters a , but it decrements counter 1 and increments counter 2. After reading $\#^2 a^k \# a^m$ the value of counter 1 is $k - m$ and counter 2 is $m - k$. In the following transition from q_3 to q_0 , the automaton terminates both counters. The aggregating function of $\mathcal{A}_{\text{diff}}$ is SUP, thus the automaton discards the lower value, i.e., the value of $\#^2 a^k \# a^m \#$ is $|k - m|$ and the automaton computes the supremum over values of all blocks. It follows that the value of $\#^2 a^{n[1]} \# a^{n[2]} \#^3 \dots$ is $\sup_{0 \leq i} |n[2 * i + 1] - n[2 * i + 2]|$.

3.2 Nested weighted automata

In this section we describe nested weighted automata introduced in [18], and closely follow the description of [18]. For more details and illustration of such automata we refer the reader to [18]. We start with an informal description.

Informal description. A nested weighted automaton consists of a labeled automaton over infinite words, called the *master automaton*, a value function f for infinite words, and a set of weighted automata over finite words, called *slave automata*. A nested weighted automaton can be viewed as follows: given a word, we consider the run of the master automaton on the word, but the weight of each transition is determined by dynamically running slave automata; and then the value of a run is obtained using the value function f . That is, the master automaton proceeds on an input word as an usual automaton, except that before it takes a transition, it can start a slave automaton corresponding to the label of the current transition. The slave automaton starts at the current position of the word of the master automaton and works on some finite part of the input word. Once a slave automaton finishes, it returns its value to the master automaton, which treats the returned value as the weight of the current transition that is being executed. Note that for some transitions the master automaton might not invoke any slave automaton, and which corresponds to *silent* transitions. If one of slave automata rejects, the nested weighted automaton rejects. We define this formally as follows.

Nested weighted automata. A *nested weighted automaton* (NWA) \mathbb{A} is a tuple $\langle \mathcal{A}_{mas}; f; \mathfrak{B}_1, \dots, \mathfrak{B}_k \rangle$, where (1) \mathcal{A}_{mas} , called the *master automaton*, is a $\{1, \dots, k\}$ -labeled automaton over infinite words (the labels are the indexes of automata $\mathfrak{B}_1, \dots, \mathfrak{B}_k$), (2) f is a value function on infinite words, called the *master value function*, and (3) $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ are weighted automata over finite words called *slave automata*. Intuitively, an NWA can be regarded as an f -automaton whose weights are dynamically computed at every step by a corresponding slave automaton. We define an $(f; g)$ -*automaton* as an NWA where the master value function is f and all slave automata are g -automata.

Semantics: runs and values. A *run* of an NWA \mathbb{A} on an infinite word w is an infinite sequence $(\Pi, \pi_1, \pi_2, \dots)$ such that (1) Π is a run of \mathcal{A}_{mas} on w ; (2) for every $i > 0$ we have π_i is a run of the automaton $\mathfrak{B}_{C(\Pi[i-1], w[i], \Pi[i])}$, referenced by the label $C(\Pi[i-1], w[i], \Pi[i])$ of the master automaton, on some finite word of $w[i, j]$. The run $(\Pi, \pi_1, \pi_2, \dots)$ is *accepting* if all runs Π, π_1, π_2, \dots are accepting (i.e., Π satisfies its acceptance condition and each π_1, π_2, \dots ends in an accepting state) and infinitely many runs of slave automata have length greater than 1 (the master automaton takes infinitely many non-silent transitions). The value of the run $(\Pi, \pi_1, \pi_2, \dots)$ is defined as $\text{sil}(f)(v(\pi_1)v(\pi_2)\dots)$, where $v(\pi_i)$ is the value of the run π_i in the corresponding slave automaton. The value of a word w assigned by the automaton \mathbb{A} , denoted by $\mathcal{L}_{\mathbb{A}}(w)$, is the infimum of the set of values of all *accepting* runs. We require accepting runs to contain infinitely many non-silent transitions because f is a value function over infinite sequences, so we need the sequence $v(\pi_1)v(\pi_2)\dots$ with \perp symbols removed to be infinite.

Deterministic nested weighted automata. An NWA \mathbb{A} is *deterministic* if (1) the master automaton and all slave automata are deterministic, and (2) slave automata recognize prefix-free languages, i.e., languages \mathcal{L} such that if $w \in \mathcal{L}$, then no proper extension of w belongs to \mathcal{L} . Condition (2) implies that no accepting run of a slave automaton visits an accepting state twice. Intuitively, slave automata have to accept the first time they encounter an accepting state as they will not see an accepting state again.

Bounded width. An NWA has *bounded width* if and only if there exists a bound C such that in every run at every position at most C slave automata are active.

Example 4 (Average response time with bounded requests). Consider an alphabet Σ consisting of requests r , grants g and null instructions $\#$. The average response time (ART) property asks for the average number of instructions between any request and the following grant. It has been shown in [18] that NWA can express ART. However, the automaton from [18] does not have bounded width. To express the ART property with NWA of bounded width we consider only words such that between any two grants there are at most k requests.

Average response time over words where between any two grants there are at most k requests can be expressed by an (LIMAVG; SUM)-automaton \mathbb{A} . Such an automaton $\mathbb{A} = (\mathcal{A}_{mas}; \text{LIMAVG}; \mathfrak{B}_1, \mathfrak{B}_2)$ is depicted in Fig. 2. The master automaton of \mathbb{A} accepts only words with infinite number of requests and grants, where every grant is followed by a request and there are at most k requests between any two grants. On letters $\#$ and g , the master automaton invokes a dummy automaton \mathfrak{B}_1 , which immediately accepts; the result of invoking such an automaton is equivalent to taking a silent transition as the automaton \mathfrak{B}_1 returns \perp , the empty value. On letters r , denoting requests, the master automaton invokes \mathfrak{B}_2 , which counts the number of letters to the first occurrence of letter g , i.e., the automaton \mathfrak{B}_2 computes the response time for the request on the position it is invoked. The automaton \mathbb{A} computes the limit average of all returned values, which is precisely ART (on the accepted words). Note that the width of \mathbb{A} is bounded by k .

3.3 Translation

We now present translations from NWA to automata with monitor counters and vice-versa.

Lemma 5. [Translation Lemma] For every value function $f \in \text{InfVal}$ on infinite words we have the following: (1) Every deterministic f -automaton with monitor counters \mathcal{A}^{m-c} can be transformed in polynomial time into an equivalent deterministic $(f; \text{SUM})$ -automaton of bounded width. (2) Every non-deterministic (resp., deterministic) $(f; \text{SUM})$ -automaton of bounded width can be transformed in exponential time into an equivalent non-deterministic (resp., deterministic) f -automaton with monitor counters.

We illustrate below the key ideas of the above translations of Lemma 5 to automata from Examples 3 and 4. The detailed technical proof is in the appendix.

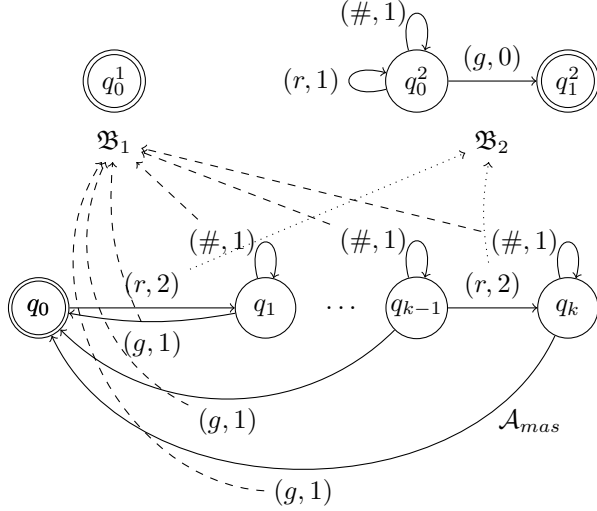


Figure 2: The (LIMAVG; SUM)-automaton computing the average response time over words with infinite number of requests and grants such that between any two grants there are at most k requests.

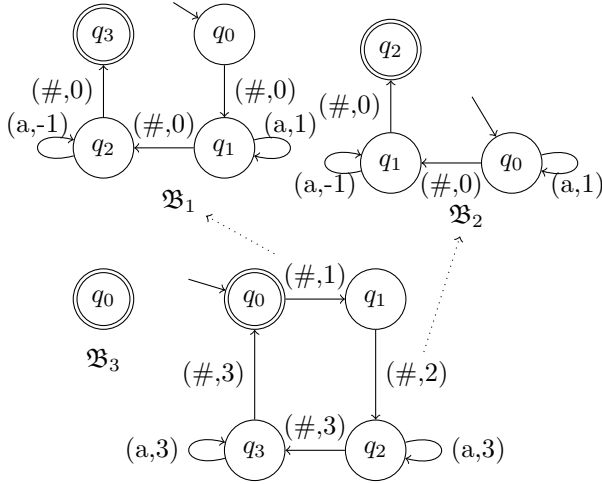


Figure 3: A nested weighted automaton resulting from translation of the automaton $\mathcal{A}_{\text{diff}}$ from Example 3.

Example 6 (Translation of automata with monitor counters to nested weighted automata). *Consider a deterministic automaton \mathcal{A} with k monitor counters. We construct an NWA \mathbb{A} equivalent to \mathcal{A} . The automaton \mathbb{A} uses k slave automata to track values of k monitor counters in the following way. The master automaton of \mathbb{A} simulates \mathcal{A} ; it invokes slave automata whenever \mathcal{A} starts monitor counters. Slave automata simulate \mathcal{A} as well. Each slave automaton is associated with some counter i ; it starts in the state (of \mathcal{A}) the counter i is initialized, simulates the value of counter i , and terminates when counter i is terminated. Figure 3 presents the result of transition of the automaton $\mathcal{A}_{\text{diff}}$ from Example 3 to a (SUP; SUM)-automaton of width bounded by 3.*

Example 7. (Translation of nested weighted automata of bounded width to automata with monitor counters) *Consider an $(f; \text{SUM})$ -automaton \mathbb{A} of width bounded by k . The automaton \mathbb{A} can be simulated by an automaton with monitor counters which simulates the master automaton and up to k slave automata running in parallel. To simulate values of slave automata it uses monitor counters, each counter separately for each slave automaton.*

Figure 4 shows the result of translation of the automaton \mathbb{A} from Example 4 to the automaton with monitor counters $\mathcal{A}_{\mathbb{A}}$. The set of states of $\mathcal{A}_{\mathbb{A}}$ there is $q_0, \dots, q_k \times (\{q_0^2, \perp\})^k$, i.e., the states of the master automaton and all non-accepting states of slave automata (in deterministic NWA accepting states

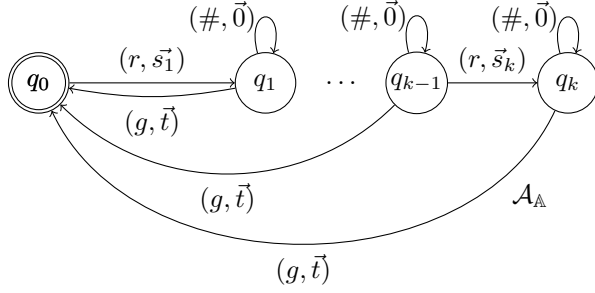


Figure 4: The (reduced) result of translation of the automaton \mathbb{A} from Example 4 to an automaton with monitor counters. Vector $\vec{0}$ (resp., \vec{t}) denotes the k -dimensional vector whose all components equal 0 (resp., t). Vector \vec{s}_i denotes the k -dimensional vector whose i -th component is s and other components are 0.

are sink states, hence storing them is redundant). Now, observe that only reachable states of $\mathcal{A}_{\mathbb{A}}$ are $(q_0, \perp, \dots, \perp), (q_1, q_0^2, \perp, \dots, \perp), \dots, (q_k, q_0^2, \dots, q_0^2)$, i.e., the reachable part of $\mathcal{A}_{\mathbb{A}}$ is isomorphic (in the sense of graphs) to the master automaton of \mathbb{A} .

Remark 8 (Discussion). Lemma 5 states that deterministic automata with monitor counters have the same expressive power as deterministic NWA of bounded width. However, the latter may be exponentially more succinct. In consequence, lower bounds on deterministic automata with monitor counters imply lower bounds on NWA of bounded width. Conversely, deterministic NWA can be considered as automata with infinite number of monitor counters, therefore upper bounds on deterministic NWA imply upper bounds on deterministic counter automata.

4 Problems

4.1 Classical questions

The classical questions in automata theory are *emptiness* and *universality* (of a language). These problems have their counterparts in the quantitative setting of weighted automata and their extensions. The (quantitative) emptiness and universality problems are defined in the same way for weighted automata, NWA and automata with monitor counters, i.e., in the following definition the automaton \mathcal{A} can be a weighted automaton, an NWA or an automaton with monitor counters.

- **Emptiness:** Given an automaton \mathcal{A} and a threshold λ , decide whether there exists a word w with $\mathcal{L}_{\mathcal{A}}(w) \leq \lambda$.
- **Universality:** Given an automaton \mathcal{A} and a threshold λ , decide whether for every word w we have $\mathcal{L}_{\mathcal{A}}(w) \leq \lambda$.

The universality question asks for *non-existence* of a word w such that $\mathcal{L}_{\mathcal{A}}(w) > \lambda$.

4.2 Probabilistic questions

The classical questions ask for the existence (or non-existence) of words for input automata, whereas in the probabilistic setting, input automata are analyzed w.r.t. a probability distribution. We consider probability distributions over infinite words Σ^ω , and as a finite representation consider the classical model of Markov chains.

Labeled Markov chains. A (labeled) Markov chain is a tuple $\langle \Sigma, S, s_0, E \rangle$, where Σ is the alphabet of letters, S is a finite set of states, s_0 is an initial state, $E : S \times \Sigma \times S \mapsto [0, 1]$ is the edge probability function, which for every $s \in S$ satisfies that $\sum_{a \in \Sigma, s' \in S} E(s, a, s') = 1$.

Distributions given by Markov chains. Consider a Markov chain \mathcal{M} . For every finite word u , the probability of u , denoted $\mathbb{P}_{\mathcal{M}}(u)$, w.r.t. the Markov chain \mathcal{M} is the sum of probabilities of paths

		INF LIMINF	SUP LIMSUP	LIMAVG
MIN, MAX SUM ^B	Empt.	PSP.-c		
	Univ.			
SUM	Empt.	PSP.-c	Undec.	Open
	Univ.	Undec.	PSP.-c	
SUM ⁺	Empt. Univ.	PSP.-c		EXPSP.

Table 1: Decidability and complexity of emptiness and universality for deterministic $(f; g)$ -automata. Functions f are listed in the first row and functions g are in the first column. PSP. denotes PSPACE, EXPSP. denotes EXPSPACE, and Undec. denotes undecidability.

labeled by u , where the probability of a path is the product of probabilities of its edges. For basic open sets $u \cdot \Sigma^\omega = \{uw : w \in \Sigma^\omega\}$, we have $\mathbb{P}_\mathcal{M}(u \cdot \Sigma^\omega) = \mathbb{P}_\mathcal{M}(u)$, and then the probability measure over infinite words defined by \mathcal{M} is the unique extension of the above measure (by Carathéodory's extension theorem [23]). We will denote the unique probability measure defined by \mathcal{M} as $\mathbb{P}_\mathcal{M}$, and the associated expectation measure as $\mathbb{E}_\mathcal{M}$.

Automata as random variables. Note that weighted automata, NWA, or automata with monitor counters all define measurable functions, $f : \Sigma^\omega \mapsto \mathbb{R}$, and thus can be interpreted as random variables w.r.t. the probabilistic space we consider. Hence given an automaton \mathcal{A} and a Markov chain \mathcal{M} , we consider the following fundamental quantities:

1. **Expected value:** $\mathbb{E}_\mathcal{M}(\mathcal{A})$ is the expected value of the random variable defined by the automaton \mathcal{A} w.r.t. the probability measure defined by the Markov chain \mathcal{M} .
2. **(Cumulative) distribution:** $\mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda) = \mathbb{P}_\mathcal{M}(\{w : \mathcal{L}_\mathcal{A}(w) \leq \lambda\})$ is the cumulative distribution function of the random variable defined by the automaton \mathcal{A} w.r.t. the probability measure defined by the Markov chain \mathcal{M} .

Computational questions. Given an automaton \mathcal{A} and a Markov chain \mathcal{M} , we consider the following basic computational questions: (Q1) The *expected question* asks to compute $\mathbb{E}_\mathcal{M}(\mathcal{A})$. (Q2) The *distribution question* asks, given a threshold λ , to compute $\mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$. Questions (1) and (2) have their approximate variants, which, given an additional input $\epsilon > 0$, ask to compute values that are ϵ -close to $\mathbb{E}_\mathcal{M}(\mathcal{A})$ or $\mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$, i.e., given $\epsilon > 0$ (Q3) The *approximate expected question* asks to compute a value η such that $|\eta - \mathbb{E}_\mathcal{M}(\mathcal{A})| \leq \epsilon$, and (Q4) The *approximate distribution question* asks to compute a value η such that $|\eta - \mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)| \leq \epsilon$. Additionally, a special important case for the distribution question is (Q5) The *almost-sure distribution question* asks whether for a given λ the probability $\mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$ is exactly 1.

We refer to questions (Q1)-(Q5) as *probabilistic questions*. Note that an upper bound on the complexity of the expected and distribution questions imply the same upper bound on all probabilistic questions as approximate and almost-sure variants are special cases.

Example 9 (Expected average response time). *Consider an NWA \mathbb{A} from Example 4. Recall that it computes ART on words it accepts (bounded number of requests between any two grants). Next, consider a Markov chain \mathcal{M} which gives a distribution on words over $\{r, g, \#\}$. In such a case, the value $\mathbb{E}_\mathcal{M}(\mathbb{A})$ is the expected ART.*

5 Results on classical questions

Existing results. The complexity of the classical decision problems for NWA has been established in [18] which is presented in Table 1.

New results. Due to Lemma 5, decidability of deterministic $(f; \text{SUM})$ -automata implies decidability of deterministic automata with monitor counters with the value function f . However, the undecidability

result of NWA does not imply undecidability for automata with monitor counters. Our following result presents the decidability picture also for automata with monitor counters (i.e., the decidability result coincides with the SUM row of Table 1).

Theorem 10. (1) *The emptiness problem is undecidable for deterministic SUP-automata (resp., LIMSUP-automata) with 8 monitor counters.* (2) *The universality problem is undecidable for deterministic INF-automata (resp., LIMINF-automata) with 8 monitor counters.*

6 Results on probabilistic questions

In this section we present our results for the probabilistic questions and deterministic NWA. First we present some basic properties, then some basic facts about Markov chains, and then present our results organized by value functions of the master automaton.

Property about almost-sure acceptance. Observe that if the probability of the set of words rejected by an automaton \mathcal{A} is strictly greater than 0, then the expected value of such an automaton is infinite or undefined. In the next lemma we show that given a deterministic NWA \mathbb{A} and a Markov chain \mathcal{M} whether the set of words accepted has probability 1 can be decided in polynomial time. In the sequel when we consider all the computational problems we consider that the set of accepted words has probability 1. This assumption does not influence the complexity of computational questions related to the expected value, but has an influence on the complexity of distribution questions, which we discuss in the appendix.

Proposition 11. *Given a deterministic NWA \mathbb{A} and a Markov chain \mathcal{M} , we can decide in polynomial time whether $\mathbb{P}_{\mathcal{M}}(\{w : \text{Acc}(w) \neq \emptyset\}) = 1$?*

6.1 Basic facts about Markov chains

Labeled Markov chains with weights. A labeled Markov chain with weights is a (labeled) Markov chain \mathcal{M} with a function r , which associates integers with edges of \mathcal{M} . Formally, a (labeled) Markov chain with weights is a tuple $\langle \Sigma, S, s_0, E, r \rangle$, where $\langle \Sigma, S, s_0, E \rangle$ is a labeled Markov chain and $r : S \times \Sigma \times S \mapsto \mathbb{Z}$.

Graph properties on Markov chains. Standard graph notions have their counterparts on Markov chains by considering edges with strictly positive probability as present and edges with probability 0 as absent. For examples, we consider the following graph notions:

- **(reachability):** A state s is *reachable* from s' in a Markov chain if there exists a sequence of edges with positive probability starting in s' and ending in s .
- **(SCCs):** A subset of states Q of a Markov chain is a *strongly connected component* (SCC) if and only if from any state of Q all states in Q are reachable.
- **(end SCCs):** An SCC Q is an *end SCC* if and only if there are no edges leaving Q .

The product of an automaton and a Markov chain. Let $\mathcal{A} = \langle \Sigma, Q, q_0, \delta, F, C \rangle$ be a deterministic weighted automaton and let $\mathcal{M} = \langle \Sigma, S, s_0, E, r \rangle$. We define the product of \mathcal{A} and \mathcal{M} , denoted by $\mathcal{A} \times \mathcal{M}$, as a Markov chain $\langle \Sigma, Q \times S, (q_0, s_0), E', r' \rangle$, where (1) $E'(\langle q_1, s_1 \rangle, a, \langle q_2, s_2 \rangle) = E(s_1, a, s_2)$ if $(q_1, a, q_2) \in \delta$ and $E'(\langle q_1, s_1 \rangle, a, \langle q_2, s_2 \rangle) = 0$ otherwise, and (2) $r'(\langle q_1, s_1 \rangle, a, \langle q_2, s_2 \rangle) = C(q_1, a, q_2) + r(s_1, a, s_2)$.

The expected value and distribution questions can be answered in polynomial time for deterministic weighted automata with value functions from InfVal [14].

Fact 12. *Let $f \in \text{InfVal}$. Given a Markov chain \mathcal{M} , a deterministic f -automaton \mathcal{A} and a value λ , the values $\mathbb{E}_{\mathcal{M}}(\mathcal{A})$ and $\mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$ can be computed in polynomial time.*

6.2 LimInf and LimSup value functions

In this section we study NWA with LIMINF and LIMSUP value functions for the master automaton. We establish polynomial-time algorithms for all probabilistic questions. We start with a result for the special case when the master automaton is strongly connected w.r.t. the Markov chain.

An automaton strongly connected on a Markov chain. We say that a deterministic automaton \mathcal{A} is *strongly connected* on a Markov chain \mathcal{M} if and only if the states reachable (with positive probability) in $\mathcal{A} \times \mathcal{M}$ from the initial state form an SCC.

Lemma 13. *Let $g \in \text{FinVal}$, \mathcal{M} be a Markov chain, and \mathbb{A} be a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{LIMINF}; g)$ -automaton). If the master automaton of \mathbb{A} is strongly connected on \mathcal{M} then there exists a unique value λ , with $|\lambda| \leq |\mathbb{A}|$ or $\lambda = -\infty$ (or $\lambda = -B$ for $g = \text{SUM}^B$), such that $\mathbb{P}_{\mathcal{M}}(\{w : \mathbb{A}(w) = \lambda\}) = 1$. Moreover, given \mathcal{M} and \mathbb{A} , the value λ can be computed in polynomial time in $|\mathcal{M}| + |\mathbb{A}|$.*

Intuitively, in the probabilistic setting, for the condition saying that a given infimal value appears infinitely often, we establish in Lemma 13 some sort of 0-1 law for SCCs which shows that a given infimum appears infinitely often either on almost all words or only on words of probability zero. Consider the product of \mathcal{M} and the master automaton of \mathbb{A} , and consider a state (s, q) in the product. Consider a slave automaton \mathfrak{B}_i that can be invoked in q , and let $\lambda_{s,q,i}$ be the minimal value that can be achieved over finite words with positive probability given that the Markov chain starts in state s . Then we establish that $\lambda = \min_{s,q,i} \lambda_{s,q,i}$, i.e., it is the minimal over all such triples. Lemma 13 implies the following main lemma of this section.

Lemma 14. *Let $g \in \text{FinVal}$. For a deterministic $(\text{LIMINF}; g)$ -automata (resp., $(\text{LIMSUP}; g)$ -automata) \mathbb{A} and a Markov chain \mathcal{M} , given a threshold λ , both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M},\mathbb{A}}(\lambda)$ can be computed in polynomial time.*

The key ideas. Consider a $(\text{LIMINF}; g)$ -automaton (resp., $(\text{LIMSUP}; g)$ -automaton) \mathbb{A} . The value $\mathbb{A}(w)$ depends only on the infinite behavior of the (unique) run of \mathbb{A} on w . Almost all runs of \mathbb{A} end up in one of the end SCCs of $\mathcal{M} \times \mathcal{A}_{mas}$, where, by Lemma 13 almost all words have the same value which can be computed in polynomial time. Thus, to compute $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$, it suffices to compute probabilities of reaching all end SCCs and values of \mathbb{A} in these components. In a similar way, we can compute $\mathbb{D}_{\mathcal{M},\mathbb{A}}(\lambda)$.

Theorem 15. *Let $g \in \text{FinVal}$. The complexity results for the probabilistic questions for $(\text{LIMINF}; g)$ -automata (resp., $(\text{LIMSUP}; g)$ -automata) are summarized in Table 2.*

	MIN, MAX, SUM ^B , SUM ⁺ , SUM
All probabilistic questions	PTime (Lemma 14)

Table 2: The complexity results for various problems for deterministic NWA with LIMSUP and LIMINF value functions.

Remark 16 (Contrast with classical questions). *Consider the results on classical questions shown in Table 1 and the results for probabilistic questions we establish in Table 2. Quite contrastingly while for classical questions the problems are PSPACE-complete or undecidable, we establish polynomial-time algorithms for all probabilistic questions.*

6.3 Inf and Sup value functions

In contrast to LIMINF and LIMSUP value functions, for which all probabilistic questions can be answered in polynomial time (Lemma 15), we first present several hardness results for INF and SUP value functions for NWA.

Lemma 17. *[Hardness results] Let $g \in \text{FinVal}$ be a value function, and \mathcal{U} denote the uniform distribution over the infinite words.*

1. The following problems are PSPACE-hard: Given a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , decide whether $\mathbb{E}_{\mathcal{U}}(\mathbb{A}) = 0$; and decide whether $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0) = 1$?
2. The following problems are $\#P$ -hard: Given $\epsilon > 0$ and a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , compute $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ up to precision ϵ ; and compute $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0)$ up to precision ϵ .

The key ideas. We present the key ideas: PSPACE-hardness. NWA have the ability to invoke multiple slave automata which independently work over the same word. In particular, one can express that the intersection of languages of finite-word automata $\mathcal{A}_1, \dots, \mathcal{A}_k$ is non-empty by tuning these automata into slave automata that return 1 if the original automaton accepts and 0 otherwise. Then, the infimum over all values of slave automata is 1 if and only if the intersection is non-empty. Note however, that words of the minimal length in the intersection can have exponential length. The probability of such word can be doubly-exponentially small in the size of \mathbb{A} , and thus the PSPACE-hardness does not apply to the approximation problems (which we establish below).

$\#P$ -hardness. We show $\#P$ -hardness of the approximate variants by reduction from $\#SAT$, which is $\#P$ -complete [31, 29]. The $\#SAT$ problem asks, given a CNF formula φ , for the number of assignments satisfying φ . In the proof, (the prefix of) the input word gives an assignment, which is processed by slave automata. Each slave automaton checks the satisfaction of one clause and return 1 if it is satisfied and 0 otherwise. Thus, all slave automata return 0 if and only if all clauses are satisfied. In such case, one can compute from $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0)$ the number of satisfying assignments of φ .

Upper bounds for $g \in \text{FinVal} \setminus \{\text{Sum}^+, \text{Sum}\}$. We now present upper bounds for value functions $g \in \text{FinVal} \setminus \{\text{Sum}^+, \text{Sum}\}$ of the slave automata. First we show an exponential-time upper bound for general NWA with INF and SUP value functions (cf. with the PSPACE-hardness from Lemma 17).

Lemma 18. *Let $g \in \text{FinVal} \setminus \{\text{Sum}^+, \text{Sum}\}$ be a value function. Given a Markov chain \mathcal{M} , a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , and a threshold λ in binary, both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time. Moreover, if \mathbb{A} has bounded width, then the above quantities can be computed in polynomial time.*

Remark 19. *We show in Lemma 18 a polynomial-time upper bound for NWA with bounded width, which gives a polynomial-time upper bound for automata with monitor counters.*

Key ideas. For $g \in \text{FinVal} \setminus \{\text{Sum}^+, \text{Sum}\}$, it has been shown in [18] that $(\text{INF}; g)$ -automata (resp., $(\text{SUP}; g)$ -automata) can be transformed to exponential-size INF-automata (resp., SUP-automata). We observe that the transformation preserves determinism. Then, using Fact 12, both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time.

The Sum^+ and Sum value functions for slave automata. We now establish the result when $g = \text{Sum}, \text{Sum}^+$. First we establish decidability of the approximation problems, and then undecidability of the exact questions.

Lemma 20. *Let $g \in \{\text{Sum}^+, \text{Sum}\}$. Given $\epsilon > 0$, a Markov chain \mathcal{M} , a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , a threshold λ , both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed up to precision ϵ in exponential time, and the dependency on ϵ is linear in the binary representation of ϵ .*

Key ideas. The main difference between INF and LIMINF value functions is that the latter discards all values encountered before the master automaton reaches an end SCC where the infimum of values of slave automata is easy to compute (Lemma 13). We show that for some B , exponential in $|\mathbb{A}|$ and polynomial in the binary representation of ϵ , the probability that any slave automaton returns value λ with $|\lambda| > B$ is smaller than ϵ . Therefore, to approximate $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ up to precision ϵ , we can regard a given $(\text{INF}; \text{Sum})$ -automaton (resp., $(\text{SUP}; \text{Sum})$ -automaton) as $(\text{INF}; \text{Sum}^B)$ -automaton (resp., $(\text{SUP}; \text{Sum}^B)$ -automaton) and use Lemma 18.

Lemma 21. *Let \mathcal{U} denote the uniform distribution over the infinite words. The following problems are undecidable: (1) Given a deterministic $(\text{INF}; \text{Sum})$ -automaton (resp., $(\text{SUP}; \text{Sum})$ -automaton) \mathbb{A} of width bounded by 8, decide whether $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(-1) = 1$. (2) Given two deterministic $(\text{INF}; \text{Sum})$ -automata (resp., $(\text{SUP}; \text{Sum})$ -automata) $\mathbb{A}_1, \mathbb{A}_2$ of width bounded by 8, decide whether $\mathbb{E}_{\mathcal{U}}(\mathbb{A}_1) = \mathbb{E}_{\mathcal{U}}(\mathbb{A}_2)$.*

Key ideas. On finite words, $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1) = 1$ holds if and only if every word has the value not exceeding -1 , i.e., the distribution question and the universality problem are equivalent. We observe that in automata from the proof of Theorem 10 such an equivalence holds as well, i.e., there exists a word with the value exceeding -1 if and only if $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1) < 1$. To show (2), we consider the automaton \mathbb{A} from (1) and its copy \mathbb{A}' that at the first transition invokes a slave automaton that returns -1 . On every word w , we have $\mathbb{A}'(w) = \min(-1, \mathbb{A}(w))$. Thus, the expected values are equal if and only if $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1) = 1$.

Finally, we have the following result for the absolute sum value function.

Lemma 22. (1) Given a Markov chain \mathcal{M} , a deterministic $(\text{INF}; \text{SUM}^+)$ -automaton \mathbb{A} , and a threshold λ in binary, both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M},\mathbb{A}}(\lambda)$ can be computed in exponential time. (2) Given a Markov chain \mathcal{M} , a deterministic $(\text{SUP}; \text{SUM}^+)$ -automaton \mathbb{A} , and a threshold λ in binary $\mathbb{D}_{\mathcal{M},\mathbb{A}}(\lambda)$ can be computed in exponential time.

The problem, how to compute $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ for deterministic $(\text{SUP}; \text{SUM}^+)$ -automata \mathbb{A} remains open.

Theorem 23. Let $g \in \text{FinVal}$. The complexity results for the probabilistic questions for $(\text{INF}; g)$ -automata and (SUP, g) -automata are summarized in Table 3, with the exception of the expected question of $(\text{SUP}; \text{SUM}^+)$ -automata.

	MIN, MAX, SUM ^B , SUM ⁺	SUM
Expected value	EXPTIME (L. 18,20) PSPACE-hard (L. 17)	Uncomputable (L. 21)
Distribution		
Almost sure distribution		
Approximate: (a) expected value (b) distribution	EXPTIME (L. 18,20) #P-hard (L. 17)	

Table 3: The complexity results for various problems for deterministic NWA with INF and SUP value functions, with exception of expected question of $(\text{SUP}, \text{SUM}^+)$ -automata which is open. Columns represent slave-automata value functions, rows represent probabilistic questions.

Open question. The decidability of the expected question of $(\text{SUP}; \text{SUM}^+)$ -automata is open. This open problem is related to the language inclusion problem of deterministic $(\text{SUP}; \text{SUM}^+)$ -automata which is also an open problem.

Remark 24 (Contrast with classical questions). Consider Table 1 for the classical questions and our results established in Table 3 for probabilistic questions. There are some contrasting results, such as, while for (SUP, SUM) -automata the emptiness problem is undecidable, the approximation problems are decidable.

Remark 25 (Contrast of LIMINF vs INF). We remark on the contrast of the LIMINF vs INF value functions. For the classical questions of emptiness and universality, the complexity and decidability always coincide for LIMINF and INF value functions for NWA (see Table 1). Surprisingly we establish that for probabilistic questions there is a substantial complexity gap: while the LIMINF problems can be solved in polynomial time, the INF problems are undecidable, PSPACE-hard, and even #P-hard for approximation.

6.4 LimAvg value function

Lemma 26. Let $g \in \text{FinVal}$. Given a Markov chain \mathcal{M} and a deterministic $(\text{LIMAVG}; g)$ -automaton \mathbb{A} , the value $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ can be computed in polynomial time.

Proof sketch. We present the most interesting case when $g = \text{SUM}$. Let \mathbb{A} be a $(\text{LIMAVG}; \text{SUM})$ -automaton and let \mathcal{M} be a Markov chain. We define a weighted Markov chain $\mathcal{M}^{\mathbb{A}}$ as the product $\mathcal{A}_{\text{mas}} \times \mathcal{M}$, where \mathcal{A}_{mas} is the master automaton of \mathbb{A} . The weights of $\mathcal{M}^{\mathbb{A}}$ are the expected values of

invoked slave automata, i.e., the weight of the transition $\langle (q, s), a, (q', s') \rangle$ is the expected value of \mathfrak{B}_i , the slave automaton started by \mathcal{A}_{mas} in the state q upon reading a , w.r.t. the distribution given by \mathcal{M} starting in s . One can show that the expected value of \mathbb{A} w.r.t. \mathcal{M} , denoted by $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$, and the expected value of $\mathcal{M}^{\mathbb{A}}$ coincide. The Markov chain $\mathcal{M}^{\mathbb{A}}$ can be computed in polynomial time and has polynomial size in $|\mathbb{A}| + |\mathcal{M}|$. Thus, we can compute the expected values of $\mathcal{M}^{\mathbb{A}}$, and in turn $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$, in polynomial time in $|\mathbb{A}| + |\mathcal{M}|$. \square

Lemma 27. *Let $g \in \text{FinVal}$. Given a Markov chain \mathcal{M} , a deterministic $(\text{LIMAVG}; g)$ -automaton \mathbb{A} and a value λ , the value $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in polynomial time.*

Key ideas. We show that the distribution is discrete. More precisely, let \mathcal{A} be the product of the Markov chain \mathcal{M} and the master automaton of \mathbb{A} . We show that almost all words, whose run end up in the same end SCC of \mathcal{A} , have the same value, which is equal to the expected value over words that end up in that SCC. Thus, to answer the distribution question, we have to compute for every end SCC C of \mathcal{A} , the expected value over words that end up in C and the probability of reaching C . Both values can be computed in polynomial time (see Lemma 26).

Theorem 28. *Let $g \in \text{FinVal}$. The complexity results for the probabilistic questions for $(\text{LIMAVG}; g)$ -automata are summarized in Table 4.*

	MIN, MAX, SUM ^B , SUM ⁺ , SUM
All probabilistic questions	PTime (Lemmas 26 and 27)

Table 4: The complexity results for various problems for deterministic NWA with LIMAVG value function.

Remark 29 (Contrast with classical questions). *Our results summarized in Table 4 contrast the results on classical questions shown in Table 1. While classical questions are PSPACE-complete, in EXPSpace or open, we establish polynomial-time algorithms for all probabilistic questions.*

7 Results on non-deterministic automata

In this section, we briefly discuss non-deterministic NWA evaluated on Markov chains. We present two negative results.

Conceptual difficulty. The evaluation of non-deterministic (even non-nested) weighted automaton over a Markov chain is conceptually different as compared to the standard model of Markov decision processes (MDPs). Indeed, in an MDP, probabilistic transitions are interleaved with non-deterministic transitions, whereas in the case of automaton, it runs over a word that has been already generated by the Markov chain. In MDPs, the strategy to resolve non-determinism can only rely on the past, whereas in the automaton model the whole future is available (i.e., there is a crucial distinction between online vs offline processing of the word). Below we present an example to illustrate the conceptual problem.

Example 30. *Consider a non-deterministic LIMAVG-automaton \mathcal{A} , depicted in Figure 5. The automaton \mathcal{A} has two states q_0, q_1 and it work over the alphabet $\Sigma = \{a, b, \#\}$. The state q_0 has transitions to itself labeled by $a, b, \#$ of weights $1, -1, 0$, respectively. The state q_1 has the same self-loops as q_0 , but weight of a is -1 and b is 1 . Also, there are transitions from q_0 to q_1 and back labeled by $\#$ of weight 0 . The automaton \mathcal{A} is depicted in Intuitively, the automaton processes a given word in blocks of letters a, b separated by letters $\#$. At the beginning of every block it decides whether the value of this block is the number of a letters n_a minus the number of b letters n_b divided by $n_a + n_b$ (i.e., $\frac{n_a - n_b}{n_a + n_b}$) or the opposite (i.e., $\frac{n_b - n_a}{n_a + n_b}$). Let \mathcal{U} be the uniform distribution on infinite words over Σ . Suppose that the expected value of \mathcal{A} w.r.t. \mathcal{U} is evaluated as in MDPs case, i.e., non-deterministic choices depend only on the read part of the word. Then, since the distribution is uniform, any strategy results in the same expected value, which is equal to 0 . Now, consider $\mathbb{E}_{\mathcal{U}}(\mathcal{A})$. The value of every block is at most 0 as the automaton works over fully generated word and at the beginning of each block can guess whether the number of a 's or b 's is greater. Also, the blocks $a\#, b\#$ with the average $-\frac{1}{2}$ appear with probability $\frac{2}{9}$, hence $\mathbb{E}_{\mathcal{U}}(\mathcal{A}) < -\frac{1}{9}$. Thus, the result of evaluating a non-deterministic weighted automaton over a Markov chain is different than evaluating it as an MDP.*

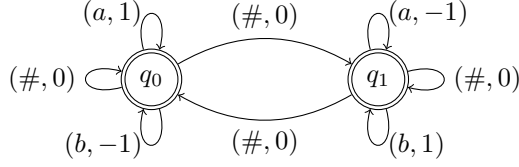


Figure 5: An example of non-deterministic automaton, in which non-deterministic choices has to “depend on the future” in order to obtain the infimum.

	Det.	Non-det.
Emptiness	Undec. (from [18])	
Probabilistic questions	PTime (L. 26)	Uncomputable (Th. 32)

Table 5: Decidability and complexity status of the classical and probabilistic questions for (LIMSUP;SUM)-automata. The negative results hold also for NWA of bounded width and automata with monitor counters.

Computational difficulty. In contrast to our polynomial-time algorithms for the probabilistic questions for deterministic NWA with (LIMSUP, SUM) value function, we establish the following undecidability result for the non-deterministic automata with width 1. Lemma 31 implies Theorem 32.

Lemma 31. *The following problem is undecidable: given a non-deterministic (LIMSUP;SUM)-automaton \mathbb{A}_M of width 1, decide whether $\mathbb{P}(\{w : \mathcal{A}_M(w) = 0\}) = 1$ or $\mathbb{P}(\{w : \mathcal{A}_M(w) = -1\}) = 1$ w.r.t. the uniform distribution $\{0, 1\}$.*

Theorem 32. *All probabilistic questions (Q1-Q5) are undecidable for non-deterministic (LIMSUP, SUM)-automata of width 1.*

8 Discussion and Conclusion

In this work we study the probabilistic questions related to NWA and automata with monitor counters. We establish the relationship between NWA and automata with monitor counters, and present a complete picture of decidability for all the probabilistic questions we consider. Our results establish a sharp contrast of the decidability and complexity of the classical questions (of emptiness and universality) and the probabilistic questions for deterministic automata (see Tables 1, 2, 3, and 4). In addition, there is also a sharp contrast for deterministic and non-deterministic automata. For example, for (LIMSUP, SUM)-automata, the classical questions are undecidable for deterministic and non-deterministic automata, while the probabilistic questions are decidable for deterministic automata, but remain undecidable for non-deterministic automata (see Table 5). We have some complexity gap (e.g., EXPTIME vs PSPACE) which is due to the fact that the computational questions we consider for Markov chains are in PTIME (as compared to NLOGSPACE for graphs), and we need to evaluate exponential-size Markov chains. Closing the complexity gap is an interesting open question.

9 Proofs

10 Equivalence of nested weighted automata and automata with monitor counters

Lemma 5. *[Translation Lemma] For every value function $f \in \text{InfVal}$ on infinite words we have the following: (1) Every deterministic f -automaton with monitor counters \mathcal{A}^{m-c} can be transformed in polynomial time into an equivalent deterministic $(f; \text{SUM})$ -automaton of bounded width. (2) Every non-deterministic (resp., deterministic) $(f; \text{SUM})$ -automaton of bounded width can be transformed in exponential time into an equivalent non-deterministic (resp., deterministic) f -automaton with monitor counters.*

Proof. (Translation of automata with monitor counters to NWA): Consider a deterministic f -automaton $\mathcal{A}^{\text{m-c}}$ with k monitor counters. We define an $(f; \text{SUM})$ -automaton \mathbb{A} , which consists of a master automaton \mathcal{A}_{mas} and slave automata $\mathfrak{B}_1, \dots, \mathfrak{B}_{k+1}$ defined as follows. The slave automaton \mathfrak{B}_{k+1} is a dummy automaton, i.e., it has only a single state which is both the initial and the accepting state. Invoking such an automaton is equivalent to taking a silent transition (with no weight). Next, the master automaton \mathcal{A}_{mas} and slave automata $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ are variants of $\mathcal{A}^{\text{m-c}}$, i.e., they share the underlying transition structure. The automaton \mathcal{A}_{mas} simulates $\mathcal{A}^{\text{m-c}}$, i.e., it has the same states and the transitions among these states as $\mathcal{A}^{\text{m-c}}$. However, whenever $\mathcal{A}^{\text{m-c}}$ activates counter i , the master automaton invokes the slave automaton \mathfrak{B}_i . The accepting condition of \mathcal{A}_{mas} is the same as of $\mathcal{A}^{\text{m-c}}$. Slave automata $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ keep track of counters $1, \dots, k$, i.e., for every $i \in \{1, \dots, k\}$, the slave automaton \mathfrak{B}_i simulates $\mathcal{A}^{\text{m-c}}$ and applies instructions of $\mathcal{A}^{\text{m-c}}$ for counter i to its value. That is, whenever $\mathcal{A}^{\text{m-c}}$ changes the value of counter i by m , the automaton \mathfrak{B}_i takes a transition of the weight m . Finally, \mathfrak{B}_i terminates precisely when $\mathcal{A}^{\text{m-c}}$ terminates counter i . The semantics of automata with monitor counters implies that \mathbb{A} accepts if and only if $\mathcal{A}^{\text{m-c}}$ accepts and, for every word, the sequences of weights produced by the runs of \mathbb{A} and $\mathcal{A}^{\text{m-c}}$ on that word coincide. Therefore, the values of \mathbb{A} and $\mathcal{A}^{\text{m-c}}$ coincide on every word.

(Translation of NWA of bounded width to automata with monitor counters): We show that non-deterministic (resp., deterministic) f -automata with monitor counters subsume non-deterministic (resp., deterministic) $(f; \text{SUM})$ -automata of bounded width. Consider a non-deterministic $(f; \text{SUM})$ -automaton \mathbb{A} with width bounded by k . We define an f -automaton $\mathcal{A}^{\text{m-c}}$ with k monitor counters that works as follows. Let Q_{mas} be the set of states of the master automaton of \mathbb{A} and Q_s be the union of the sets of states of the slave automata of \mathbb{A} . The set of states of $\mathcal{A}^{\text{m-c}}$ is $Q_{\text{mas}} \times Q_s \times \dots \times Q_s = Q_{\text{mas}} \times (Q_s)^k$. The automaton $\mathcal{A}^{\text{m-c}}$ simulates runs of the master automaton and slave automata by keeping track of the state of the master automaton and states of up to k active slave automata. Moreover, it uses counters to simulate the values of slave automata, i.e., whenever a slave automaton is activated, $\mathcal{A}^{\text{m-c}}$ simulates the execution of this automaton and assigns some counter i to that automaton. Next, when the simulated slave automaton takes a transition of the weight m the automaton $\mathcal{A}^{\text{m-c}}$ changes the value of counter i by m . Finally, $\mathcal{A}^{\text{m-c}}$ terminates counter i when the corresponding slave automaton terminates.

Since \mathbb{A} has width bounded by k , the simulating automaton $\mathcal{A}^{\text{m-c}}$ never runs out of counters to simulate slave automata. Moreover, as it simulates runs of the master automaton and slave automata of \mathbb{A} , there is a one-to-one correspondence between runs of $\mathcal{A}^{\text{m-c}}$ and runs of \mathbb{A} and accepting runs of \mathbb{A} correspond to accepting runs of $\mathcal{A}^{\text{m-c}}$. Finally, the sequence of weights for the master automaton determined by a given run of \mathbb{A} coincides with the sequence of weights of $\mathcal{A}^{\text{m-c}}$ on the corresponding run. Therefore, the values of \mathbb{A} and $\mathcal{A}^{\text{m-c}}$ coincide on every word. Thus, non-deterministic f -automata with monitor counters subsume non-deterministic $(f; \text{SUM})$ -automata of bounded width. Moreover, the one-to-one correspondence between runs of \mathbb{A} and $\mathcal{A}^{\text{m-c}}$ implies that if \mathbb{A} is deterministic, then $\mathcal{A}^{\text{m-c}}$ is deterministic. Therefore, deterministic f -automata with monitor counters subsume deterministic $(f; \text{SUM})$ -automata of bounded width. This completes the proof. \square

11 Basic properties

Almost-sure acceptance for deterministic NWA. We present the proof of Proposition 11.

Proposition 11. *Given a deterministic NWA \mathbb{A} and a Markov chain \mathcal{M} , we can decide in polynomial time whether $\mathbb{P}_{\mathcal{M}}(\{w : \text{Acc}(w) \neq \emptyset\}) = 1$?*

Proof. The master automaton has to accept almost all words. For all pairs (q, s) , where q is the initial state of some slave automaton \mathfrak{B}_i and s is a state of the Markov chain \mathcal{M} , we check that either \mathfrak{B}_i is almost-surely not invoked while \mathcal{M} is in the state s or \mathfrak{B}_i almost-surely accepts (w.r.t. the distribution given by \mathcal{M} started in s). One can easily check that this condition is necessary and sufficient, and it can be checked in polynomial time. \square

Remark 33 (Almost-sure acceptance). *In the main article we consider that we have almost-sure acceptance. As mentioned (in paragraph “Property of almost-sure acceptance” in before Section 6.2), the answer to the expected value problem does not change even without the assumption. We will show next that without the almost-sure acceptance condition, the distribution questions become similar to INF and*

SUP value functions. Hence in the main article we consider the almost-sure acceptance property, and presented the conceptually interesting results. Moreover, classically weighted automata have been considered without any acceptance conditions (i.e., all words are accepted), and then the almost-sure acceptance is trivially ensured.

Non almost-sure acceptance. We show that removing the restriction that almost all words are accepted changes the complexity of distribution questions. The intuition behind this is that the condition “all slave automata accept” allows to simulate (in a restricted way) INF value function. This also indicates that the assumption that almost all words are accepting is justified in the probabilistic framework.

Lemma 34. *Assume that the set of rejected words can have non-zero probability. Then, for all $f \in \text{InfVal}$ and $g \in \text{FinVal}$, we have*

1. *The distribution question for deterministic $(f; g)$ -automata is PSPACE-hard.*
2. *The approximate distribution question for deterministic $(f; g)$ -automata is $\#P$ -hard.*

Proof. The distribution (resp., approximate distribution) question for deterministic $(\text{INF}; g)$ -automata with slaves returning only values 0, 1 reduce to the distribution (resp., approximate distribution) question for $(f; g)$ -automata. Given a deterministic $(\text{INF}; g)$ -automaton \mathbb{A} we modify it to \mathbb{A}^f by modifying all slave automata so that each slave automaton instead of returning 1, rejects. Observe that if a deterministic g -automaton returns only values 0, 1, then the set of words with value 1 is regular. Now, for every word w , the following conditions are equivalent

- $\mathbb{A}^f(w)$ accepts,
- $\mathbb{A}^f(w) = 0$, and
- $\mathbb{A}(w) = 0$.

By Lemma 17, the distribution question for deterministic $(\text{INF}; g)$ -automata is PSPACE-hard and the approximate distribution question for deterministic $(\text{INF}; g)$ -automata is $\#P$ -hard. Observe that slave automata in the proof of Lemma 17 return values 0, 1. Hence, the result follows. \square

In the sequel of the appendix, we present the conceptually interesting results with the almost-sure acceptance condition. As mentioned above, in the classical setting of weighted automata which has no accepting condition, the almost-sure acceptance is trivially satisfied. Before the other results, we present a technical duality result, which will be used in the proofs.

Duality property between infimum and supremum. In the sequel, when we consider the expected value and the distribution, in most cases we consider only INF and LIMINF value functions, and by duality, we obtain results for SUP and LIMSUP value functions, respectively. The only exception are $(\text{INF}, \text{SUM}^+)$ -automata and $(\text{SUP}, \text{SUM}^+)$ -automata, which have to be considered separately. For every value function $g \in \text{FinVal} \setminus \{\text{SUM}^+\}$ we define $-g$ as follows: $-\text{MIN} = \text{MAX}$, $-\text{MAX} = \text{MIN}$ and $-g = g$ for $g \in \{\text{SUM}^B, \text{SUM}\}$.

Lemma 35. *For every $g \in \text{FinVal} \setminus \{\text{SUM}^+\}$, every deterministic $(\text{SUP}; g)$ -automaton (resp. $(\text{LIMSUP}; g)$ -automaton) \mathbb{A}_1 can be transformed to a deterministic $(\text{INF}; -g)$ -automaton (resp. $(\text{LIMINF}; -g)$ -automaton) \mathbb{A}_2 of the same size such that for every word w we have $\mathbb{A}_1(w) = -\mathbb{A}_2(w)$.*

Proof. The automaton \mathbb{A}_2 is obtained from \mathbb{A}_1 by multiplying all the weights by -1 . \square

12 Undecidability results

In this section we prove Theorem 10, Lemma 21 and Theorem 31. They have similar proofs, therefore we give the proof of Theorem 10 in detail and for the remaining two results we only discuss the changes needed to adapt the proof of Theorem 10 to show these results.

Theorem 10. (1) *The emptiness problem is undecidable for deterministic SUP-automata (resp., LIMSUP-automata) with 8 monitor counters.* (2) *The universality problem is undecidable for deterministic INF-automata (resp., LIMINF-automata) with 8 monitor counters.*

Proof. We show undecidability of the emptiness problem for deterministic LIMSUP-automata with 8 monitor counters. The proof for deterministic SUP-automata is virtually the same. Then, the reduction of the universality problem for deterministic INF-automata (resp., LIMINF-automata) to the emptiness problem

for deterministic SUP-automata (resp., LIMSUP-automata) follows from the converse of Lemma 35. Intuitively, it suffices to multiply all weights in a deterministic INF-automaton (resp., LIMINF-automaton) by -1 .

The halting problem for deterministic two-counter machines is undecidable [27]. Let \mathcal{M} be a deterministic two-counter machine and let Q be the set of states of \mathcal{M} . We define a deterministic LIMSUP-automaton \mathcal{A} with 8 monitor counters such that \mathcal{A} has a run of the value not exceeding 0 if and only if \mathcal{M} has an accepting computation.

Consider the alphabet $\Sigma = Q \cup \{1, 2, \#, \$\}$. We encode computations of \mathcal{M} as a sequence of configurations separated by $\#$. A single configuration of \mathcal{M} , where the machine is in the state q , the first counter has the value x and the second y is encoded by the word $q1^x2^y$. Finally, computations of \mathcal{M} are separated by $\$$. We define the automaton \mathcal{A} that for a word $w \in \Sigma^*$ returns the value 0 if (some infinite suffix of) w encodes a sequence valid accepting computations of \mathcal{M} . Otherwise, \mathcal{A} returns the value at least 1.

The automaton \mathcal{A} works as follows. On a single computation, i.e., between symbols $\$$, it checks consistency of the transitions by checking two conditions: (C1) Boolean consistency, and (C2) counter consistency. The condition (C1) states that encoded subsequence configurations, which are represented by subwords $q1^x2^y\#q'1^{x'}2^{y'}$, are consistent with the transition function of \mathcal{M} modulo counter values, i.e., under counter abstraction to values 0 and “strictly positive”. Observe that a finite automaton can check that. The conditions that need to be checked are as follows: (C1-1) Boolean parts of transitions are consistent; the automaton checks only emptiness/nonemptiness of counters and based on that verifies whether a subword $q1^x2^y\#q'$ is valid w.r.t. transitions of \mathcal{M} . For example, consider transition $(q, \perp, +, q', +1, -1)$ of \mathcal{M} stating that “if \mathcal{M} is in state q , the first counter is 0 and the second counter is positive, then change the state to q' increment the first counter and decrement the second one”. This transition corresponds to the regular expression $q2^+\#q'$. (C1-2) The initial and finite configurations in each computation (between $\$$ symbols) are respectively $q_I1^{02^0}$ and $q_F1^{02^0}$. (C1-3) The word encodes infinitely many computations, i.e., the word contains infinitely many $\$$ symbols. The last conditions rejects words encoding non-terminating computations.

To check the condition (C2), \mathcal{A} uses monitor counters. It uses 4 monitor counters to check transitions between even and odd positions and the remaining 4 to check validity of the remaining transitions. Then, between even and odd positions it uses 2 monitor counters for each counter of \mathcal{M} . These monitor counters encode the absolute values between the intended values of counters (i.e., assuming that counter values are consistent with the instructions) and the actual values. For example, for a subword $q1^x2^y\#q'1^{x'}2^{y'}$, the automaton \mathcal{A} checks whether the value of counter 1 is consistent with transition $(q, \perp, +, q', +1, -1)$ in the following way. First monitor counter ignores letters 2 and initially decrements its value at every letter 1 until it reads letter $\#$ (where its value is $-x$). Next, it switches its mode and increments its value at letters 1 while ignoring letters 2. In that way its value upon reading $q1^x2^y\#q'1^{x'}2^{y'}$ equals $-x + x'$. Finally, it increments once counter 1. Thus, the value of the monitor counter 1 is $-x + x' + 1$. The second monitor counter works in a similar way, but it decrements whenever the first counter increments and vice versa. At the end, the value of the second monitor counter is $x - x' - 1$. Observe that the sup of them is $|x - (x' + 1)|$, which is 0 if and only if the value of counter 1 is consistent with the transition $(q, \perp, +, q', +1, -1)$. It follows that the supremum over the values of all counters is 0 only if all counter values are consistent with the transitions. Therefore, the value LIMSUP of the whole word is 0 if and only if starting at some point all computations are valid and accepting. The latter is possible only if \mathcal{M} has at least one such a computation. Otherwise, the value of LIMSUP is at least 1. \square

Lemma 21. *Let \mathcal{U} denote the uniform distribution over the infinite words. The following problems are undecidable: (1) Given a deterministic (INF;SUM)-automaton (resp., (SUP;SUM)-automaton) \mathbb{A} of width bounded by 8, decide whether $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(-1) = 1$. (2) Given two deterministic (INF;SUM)-automata (resp., (SUP;SUM)-automata) $\mathbb{A}_1, \mathbb{A}_2$ of width bounded by 8, decide whether $\mathbb{E}_{\mathcal{U}}(\mathbb{A}_1) = \mathbb{E}_{\mathcal{U}}(\mathbb{A}_2)$.*

Proof. (1): In the following, we discuss how to adapt the proof of Theorem 10 to prove this lemma.

Given a deterministic two-counter machine \mathcal{M} , we construct a deterministic (INF;SUM)-automaton $\mathbb{A}_{\mathcal{M}}$ such that for a word w of the form $\$u\w' it returns 0 if u is a valid accepting computation of \mathcal{M} and a negative values otherwise. We use $\Sigma = Q \cup \{1, 2, \#, \$\}$ for convenience; one can encode letters from Σ using two-letter alphabet $\{0, 1\}$. On words that are not of the form $\$u\w' , the automaton $\mathbb{A}_{\mathcal{M}}$ returns -1 . Basically, the automaton $\mathbb{A}_{\mathcal{M}}$ simulates on u the execution of \mathcal{A} (as defined in the proof of

Theorem 10)) with the opposite values of counters, i.e., if a counter of \mathcal{A} has a value k , the corresponding counter in the automaton we simulate stores $-k$. The simulation is virtually the same as in Lemma 5. Recall, that the supremum of monitor counters at a subword $\$u\$$ is 0 if and only if u encodes valid and accepting computation of \mathcal{M} . Otherwise, the supremum is at least 1. Thus, in our case, the infimum over the values of slave automata is 0 if and only if u encodes a valid and accepting computation of \mathcal{M} . Otherwise, the value of INF is at most -1 . Therefore, $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1) = 1$ if and only if \mathcal{M} does not have an accepting computation.

(2): We show that knowing how to compute the expected value of deterministic (INF; SUM)-automata, we can decide equality in the distribution question. Let \mathbb{A} be an automaton and we ask whether $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1) = 1$. We construct another automaton \mathbb{A}' that simulates \mathbb{A} but at the first transition invokes a slave automaton that returns the value -1 . The values of automata \mathbb{A} and \mathbb{A}' differ precisely on words which have values (assigned by \mathbb{A}) greater than -1 . Thus, their expected values $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ and $\mathbb{E}_{\mathcal{U}}(\mathbb{A}')$ differ if and only if $\mathbb{D}_{\mathcal{U},\mathbb{A}}(-1)$ is different than 1. Due to undecidability of the latter problem, there is no terminating Turing machine that computes the expected value of (INF; SUM)-automata over the uniform distribution. \square

Lemma 31. *The following problem is undecidable: given a non-deterministic (LIMSUP; SUM)-automaton \mathbb{A}_M of width 1, decide whether $\mathbb{P}(\{w : \mathcal{A}_M(w) = 0\}) = 1$ or $\mathbb{P}(\{w : \mathcal{A}_M(w) = -1\}) = 1$ w.r.t. the uniform distribution $\{0, 1\}$.*

Proof. In the following, we discuss how to adapt the proof of Theorem 10 to prove this lemma. First, observe that we can encode any alphabet Σ using two-letter alphabet $\{0, 1\}$, therefore we will present our argument for multiple-letters alphabet as it is more convenient. Given a two-counter machine \mathcal{M} we construct a non-deterministic (LIMSUP; SUM)-automaton \mathbb{A}_M such that $\mathcal{A}_M(w) = 0$ if and only if w contains infinitely many subsequences that correspond to valid accepting computations of \mathcal{M} . As in (1), for every subsequence $\$u\$$, where u does not contain $\$$, we check whether u is an encoding of a valid accepting computation of \mathcal{M} . To do that, we check conditions (C1) and (C2) as in the proof of Theorem 10, but using slave automata. At the letter $\$$, the master automaton non-deterministically decides whether u violates (C1) or (C2) and either starts a slave automaton checking (C1) or (C2). The slave automaton checking (C1) works as in the proof of Theorem 10. It returns -1 if (C1) is violated and 0 otherwise. The slave automaton checking (C2) non-deterministically picks the position of inconsistency and one of the monitor counter from the proof of Theorem 10 that would return a negative value. The slave automaton simulates this monitor counter. Finally, at the letter $\$$ following u , the master automaton starts the slave automaton that returns the value -1 . It follows that the supremum of all values of slave automata started at $u\$$ is either -1 or 0. By the construction, there is a (sub) run on $u\$$ such that the supremum of the values of all slave automata is -1 if and only if u does not encode valid accepting computation of \mathcal{M} . Otherwise, this supremum is 0. Therefore, the value of the word w is 0 if and only if w contains infinitely many subsequences that correspond to valid accepting computations of \mathcal{M} . Now, if \mathcal{M} has at least one valid accepting computation u , then almost all words contain infinitely many occurrences of u and almost all words have value 0. Otherwise, all words have value -1 . This implies that there is no terminating Turing machine that computes any of probabilistic questions. \square

13 Proofs from Section 6.2

Lemma 13. *Let $g \in \text{FinVal}$, \mathcal{M} be a Markov chain, and \mathbb{A} be a deterministic (INF; g)-automaton (resp., (LIMINF; g)-automaton). If the master automaton of \mathbb{A} is strongly connected on \mathcal{M} then there exists a unique value λ , with $|\lambda| \leq |\mathbb{A}|$ or $\lambda = -\infty$ (or $\lambda = -B$ for $g = \text{SUM}^B$), such that $\mathbb{P}_{\mathcal{M}}(\{w : \mathbb{A}(w) = \lambda\}) = 1$. Moreover, given \mathcal{M} and \mathbb{A} , the value λ can be computed in polynomial time in $|\mathcal{M}| + |\mathbb{A}|$.*

Proof. In a nutshell, we exploit the fact that the set of words in which all finite subwords occur infinitely often has probability 1. Therefore, every slave automaton (which is invoked infinitely often) runs on every finite word infinitely often. In particular, every slave automaton runs infinitely often on the words that correspond to runs of the minimal value.

More precisely, since \mathbb{A} is deterministic, all runs of \mathbb{A} on the distribution given by \mathcal{M} correspond to the runs in $\mathcal{A}_{mas} \times \mathcal{M}$, where \mathcal{A}_{mas} is the master automaton of \mathbb{A} . Among runs in $\mathcal{A}_{mas} \times \mathcal{M}$, the set of

runs where a given finite sequence of states occurs finitely many times has probability 0. In particular, for every state (q, s) , we consider a sequence that corresponds to the master automaton invoking some slave \mathfrak{B}_i followed by a word on which \mathfrak{B}_i return its minimal value. Therefore, almost all runs contain the considered sequence infinitely often. It follows that the value of almost all runs is the minimum over reachable states (q, s) from $\mathcal{A}_{mas} \times \mathcal{M}$ and transitions (s, a, s') of \mathcal{M} of the minimal value the slave automaton invoked in (q, a, s') can achieve on all words generated by \mathcal{M} starting with the transition (s, a, s') . Such values can be computed in polynomial time in $|\mathcal{M}| + |\mathbb{A}|$. Each of these values is either $-\infty$ (or $-B$ for $g = \text{SUM}^B$) or it is the sum of some subset of weights in some slave automaton. Since we consider weights to be given in the unary notation, the sum of a subset is bounded by the size of the automaton. Thus, $|\lambda| \leq |\mathbb{A}|$ or $\lambda = -\infty$ (resp., $-B$ for $g = \text{SUM}^B$). \square

Lemma 14. *Let $g \in \text{FinVal}$. For a deterministic $(\text{LIMINF}; g)$ -automata (resp., $(\text{LIMSUP}; g)$ -automata) \mathbb{A} and a Markov chain \mathcal{M} , given a threshold λ , both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in polynomial time.*

Proof. First, we discuss how to compute the expected and the distribution questions of a deterministic $(\text{LIMINF}; \text{SUM})$ -automaton \mathbb{A} .

The value of $(\text{LIMINF}; \text{SUM})$ -automaton \mathbb{A} on a word depends on weights that appear infinitely often. Since \mathbb{A} reaches some end SCC with probability 1, we can neglect values of slave automata returned before the master automaton \mathcal{A}_{mas} (of \mathbb{A}) reaches an end SCC of $\mathcal{A}_{mas} \times \mathcal{M}$. Thus, the expected value of $(\text{LIMINF}; \text{SUM})$ -automaton \mathbb{A} w.r.t. a Markov chain \mathcal{M} can be computed in the following way. Let S_1, \dots, S_l be all end SCCs of $\mathcal{A}_{mas} \times \mathcal{M}$. We compute probabilities p_1, \dots, p_l of reaching the components S_1, \dots, S_l respectively. These probabilities can be computed in polynomial time. Next, for every component S_i we compute in polynomial time the unique value m_i , which \mathbb{A} returns on almost every word whose run ends up in S_i (Lemma 13). The expected value $\mathbb{E}_{\mathcal{M}, \mathbb{A}}$ is equal to $p_1 \cdot m_1 + \dots + p_l \cdot m_l$. Observe that, given a value λ , the distribution $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ is equal to the sum the probabilities p_i over such i that $m_i \leq \lambda$. Hence, the expected and the distribution questions can be computed in polynomial time.

The remaining probabilistic questions are special cases of the expected and the distribution questions. Due to Lemma 35, the case of LIMSUP reduces to the case of LIMINF . All value functions from FinVal are special cases of SUM . This concludes the proof. \square

14 The proofs from Section 6.3

14.1 Hardness results

Lemma 17. *[Hardness results] Let $g \in \text{FinVal}$ be a value function, and \mathcal{U} denote the uniform distribution over the infinite words.*

1. *The following problems are PSPACE-hard: Given a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , decide whether $\mathbb{E}_{\mathcal{U}}(\mathbb{A}) = 0$; and decide whether $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0) = 1$?*
2. *The following problems are #P-hard: Given $\epsilon > 0$ and a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , compute $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ up to precision ϵ ; and compute $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0)$ up to precision ϵ .*

Proof. We present the following argument for $g = \text{MIN}$ and the proof for works for $g = \text{MAX}$. Lemma 35 implies that problems in (i) and (ii) for nested weighted automaton with SUP value function reduce to the corresponding problems for nested weighted automaton with INF value function. Since, MIN can be regarded as a special case of SUM^B , SUM^+ or SUM , the result holds for these functions as well. Hence, we consider only the case of (INF, MIN) -automata. **PSPACE-hardness:** We show PSPACE-hardness by reduction from the emptiness problem for the intersection of regular languages. Let $\mathcal{L}_1, \dots, \mathcal{L}_n \subseteq \{a, b\}^*$ be regular languages recognized by deterministic finite automata $\mathcal{A}_1, \dots, \mathcal{A}_n$. We define a deterministic $(\text{INF}; \text{MIN})$ -automaton \mathbb{A} that at first n steps starts slave automata $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ and then it invokes only a dummy slave automaton that returns 1 after a single step. For every i , the slave automaton \mathfrak{B}_i first reads $n - i$ letters which it ignores, then it simulates \mathcal{A}_i until the first $\#$ when it terminates. It returns 1 if the simulated automaton \mathcal{A}_i accepts and 0 otherwise. More precisely, \mathfrak{B}_i works on subwords $uv\#$, where $u \in \{a, b, \#\}^{n-i}$, $v \in \{a, b\}^*$ and returns 1 if $v \in \mathcal{L}_i$ and 0 otherwise. Observe that on a word $w = uv\#w'$ where $u \in \{a, b, \#\}^{n-i}$, $v \in \{a, b\}^*$ and $w' \in \{a, b, \#\}^\omega$, the automaton \mathbb{A} returns 1 if and only

if all automata $\mathcal{A}_1, \dots, \mathcal{A}_n$ accept v . Otherwise, \mathbb{A} assigns value 0 to w . In consequence, the following conditions are equivalent: (1) the intersection $\mathcal{L}_1 \cap \dots \cap \mathcal{L}_n$ is empty, (2) the expected value $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ is 0, and (3) the distribution $\mathbb{D}_{\mathcal{U}, \mathbb{A}}(0) = 1$.

Note that the almost-sure distribution question is PSPACE-hard as well.

Observe that if the intersection $\mathcal{L}_1 \cap \dots \cap \mathcal{L}_n$ is non-empty it might be the case that the word of the minimal length in the intersection consists of a single word of exponential length. In such a case, the values $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ and $|1 - \mathbb{D}_{\mathcal{U}, \mathbb{A}}(0)|$ are non-zero, but doubly-exponentially small. Therefore, we cannot use this reduction to show hardness of the approximate versions of the probabilistic problems.

#P-hardness: We show #P-hardness by reduction from the #SAT, which, given a propositional formula φ in conjunctive normal form asks for the number of valuations that satisfy φ . Let n be the number of variables of φ and let C_1, \dots, C_m be the clauses of φ . For every $i \in [1, m]$, we define a slave automaton \mathfrak{B}_i (associated with C_i) that ignores first $m - i$ letters, next considers the following n letters 0, 1 as the valuation of the successive variables and checks whether this valuation satisfies the clause C_i . If it does, the slave automaton returns 1, otherwise it returns 0. The master automaton first invokes slave automata $\mathfrak{B}_1, \dots, \mathfrak{B}_m$ and then it invokes a dummy slave automaton that returns 1 after a single step. Observe that for $w = uvw'$, where $u \in \{0, 1\}^m, v \in \{0, 1\}^n$ and $w' \in \{0, 1\}^\omega$, the automaton \mathbb{A} returns 1 on w if and only if the valuation given by v satisfies all clauses C_1, \dots, C_m , i.e., it satisfies φ . Otherwise, \mathbb{A} returns 0 on w . Therefore, the following values are equal and multiplied by 2^n give the number of valuations satisfying φ : $\mathbb{E}_{\mathcal{U}}(\mathbb{A})$ and $1 - \mathbb{D}_{\mathcal{U}, \mathbb{A}}(0)$. Therefore, all approximate probabilistic questions are #P-hard. \square

14.2 The upper bound for $g \in \text{FinVal} \setminus \{\text{Sum}\}$

Overview. First, we show the translation lemma (Lemma 36), which states that deterministic $(\text{INF}; \text{SUM}^B)$ -automata can be translated to deterministic INF-automata with exponential blow-up. Moreover, this blow-up can be avoided by considering NWA of bounded width and B given in unary. Since the probabilistic questions can be solved for INF-automaton in polynomial time, Lemma 36 implies that all probabilistic questions can be solved in exponential time for deterministic $(\text{INF}; \text{SUM}^B)$ -automata.

Lemma 36. (1) Given $B > 0$ in the binary notation and a deterministic $(\text{INF}; \text{SUM}^B)$ -automaton \mathbb{A} , one can construct in exponential time an exponential-size deterministic INF-automaton \mathcal{A} such that for every word w we have $\mathbb{A}(w) = \mathcal{A}(w)$. (2) Let $k > 0$. Given $B > 0$ in the unary notation and a deterministic $(\text{INF}; \text{SUM}^B)$ -automaton \mathbb{A} of width bounded by k , one can construct in polynomial time a polynomial-size deterministic INF-automaton \mathcal{A} such that for every word w we have $\mathbb{A}(w) = \mathcal{A}(w)$.

Proof. (1): Let Q_m be the set of states of the master automaton and Q_s be the union of the sets of states of slave automata of \mathbb{A} . We define an INF-automaton \mathcal{A} over the set of states $Q_m \times (Q_s \times [-B, B] \cup \{\perp\})^{|Q_s|}$. Intuitively, \mathcal{A} simulates runs of \mathbb{A} by simulating (a) the run of the master automaton using the component Q_m and (b) selected runs of up to $|Q_s|$ slave automata using the component $(Q_s \times [-B, B])^{|Q_s|}$. Slave automata are simulated along with their values, which are stored in the state, i.e., the state (q, l) encodes that a given slave automaton is in the state q and its current value is l . Then, the value of a given transition of \mathcal{A} is the minimum over the values of simulated slave automata that terminate at the current step. Finally, the symbol \perp denotes “free” components in the product $(Q_s \times [-B, B] \cup \{\perp\})^{|Q_s|}$, which can be used to simulate newly invoked slave automata. We need to convince ourselves that we need to simulate at most $|Q_s|$ slave automata. Therefore, every time a new slave automaton is invoked, we have a free component to simulate it.

Observe that if at some position two slave automata $\mathfrak{B}_1, \mathfrak{B}_2$ are in the same state q and they have collected partial values $l_1 \leq l_2$, then we can discard the simulation of the automaton \mathfrak{B}_2 , which collected the value l_2 . Indeed, since slave automata are deterministic and recognize prefix free languages, the remaining runs of both slave automata $\mathfrak{B}_1, \mathfrak{B}_2$ are the same, i.e., they either both reject or both return values, respectively, $l_1 + v$ and $l_2 + v$ for some common v . Thus, the run of \mathfrak{B}_2 does not change the value of the infimum and we can stop simulating it, i.e., we can substitute (q, l_2) by \perp . Therefore, at every position at most $|Q_s|$ components are necessary. It follows from the construction that the values of \mathbb{A} and \mathcal{A} coincide on every word.

(2): If B is given in the unary notation and the width is bounded by k , we can basically repeat the construction as above for the automaton with the set of states $Q_m \times (Q_s \times [-B, B] \cup \{\perp\})^k$, which is

polynomial in \mathbb{A} . Thus, the result of such construction is of polynomial size and can be constructed in polynomial time. \square

Lemma 18. *Let $g \in \text{FinVal} \setminus \{\text{SUM}^+, \text{SUM}\}$ be a value function. Given a Markov chain \mathcal{M} , a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , and a threshold λ in binary, both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time. Moreover, if \mathbb{A} has bounded width, then the above quantities can be computed in polynomial time.*

Remark 37. *For $g = \text{SUM}^B$, the value B given in binary can be a part of input.*

We first prove Lemma 18 for $g \in \text{FinVal} \setminus \{\text{SUM}, \text{SUM}^+\}$. Next, in Lemma 38 we show that Lemma 18 for deterministic $(\text{INF}; \text{SUM}^+)$ -automata. The statement of Lemma 38 is more general though.

Proof. Observe that deterministic weighted automata with MIN and MAX values functions can be transformed in polynomial time to deterministic weighted automata with SUM^B value function. Basically, a deterministic SUM^B -automaton simulating an MIN-automaton (resp., MAX-automaton) takes transitions of weight 0 and stores in its states the current minimal (resp., maximal) weight. Its final transition has the weight equal to the minimal (resp., maximal) weight encountered among the run. Such a SUM^B -automaton computes on every word the same value as the given MIN-automaton (resp., MAX-automaton). Therefore, we need to focus on $g = \text{SUM}^B$. Consider a deterministic $(\text{INF}; \text{SUM}^B)$ -automaton \mathbb{A} . By Lemma 36, \mathbb{A} can be transformed in exponential time into an exponential-size deterministic INF-automaton \mathcal{A} such that for every word w we have $\mathbb{A}(w) = \mathcal{A}(w)$. It follows that for every Markov chain \mathcal{M} over the alphabet of \mathbb{A} and every value λ we have $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}_{\mathcal{M}}(\mathcal{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda) = \mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$. The values $\mathbb{E}_{\mathcal{M}}(\mathcal{A}), \mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$ can be computed in polynomial time in \mathcal{A} (Fact 12), which amounts to exponential time in \mathbb{A} . Observe, however, that for \mathbb{A} of bounded width the automaton \mathcal{A} has polynomial size (assuming that the bound on the width is constant), and the values $\mathbb{E}_{\mathcal{M}}(\mathcal{A}), \mathbb{D}_{\mathcal{M}, \mathcal{A}}(\lambda)$ can be computed in polynomial time in \mathbb{A} . \square

Now, we turn to deterministic $(\text{INF}; \text{SUM})$ -automata. First, we show that under additional assumptions on slave automata, the probabilistic questions can be computed.

Lemma 38. *Given a Markov chain \mathcal{M} , a value λ and a deterministic $(\text{INF}; \text{SUM})$ -automaton such that the value of every slave automaton is bounded from below, the values $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time.*

Proof. Consider a deterministic $(\text{INF}; \text{SUM})$ -automaton \mathbb{A} such that the value of every slave automaton is bounded from below. Let $B = |\mathbb{A}|$ and let \mathbb{A}' be \mathbb{A} considered as a deterministic $(\text{INF}; \text{SUM}^B)$ -automaton. We show that on almost all words w we have $\mathbb{A}(w) = \mathbb{A}'(w)$. Then, $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}_{\mathcal{M}}(\mathbb{A}')$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda) = \mathbb{D}_{\mathcal{M}, \mathbb{A}'}(\lambda)$ and the values $\mathbb{E}_{\mathcal{M}}(\mathbb{A}')$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}'}(\lambda)$ can be computed in exponential time by Lemma 18 taking into account the remark about B being input.

Since the value of every slave automaton \mathfrak{B}_i is bounded from below, the (reachable part of) automaton \mathfrak{B}_i considered as a weighted graph does not have negative cycles. Therefore, the minimal value \mathfrak{B}_i can achieve is greater than $|\mathfrak{B}_i| < |\mathbb{A}|$. Moreover, every run of \mathbb{A} ends up in some SCC of $\mathcal{A}_{\text{mas}} \times \mathcal{M}$, where almost all words have the same value (Lemma 13), which is bounded from above by $|\mathbb{A}|$ and can be computed in polynomial time. Therefore, the value of almost all words belong to the interval $[-|\mathbb{A}|, |\mathbb{A}|]$. \square

The above lemma implies that the probabilistic questions for deterministic $(\text{INF}; \text{SUM}^+)$ -automata can be answered in exponential time, i.e., we have showed (1) from the following lemma.

Lemma 22. *(1) Given a Markov chain \mathcal{M} , a deterministic $(\text{INF}; \text{SUM}^+)$ -automaton \mathbb{A} , and a threshold λ in binary, both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time. (2) Given a Markov chain \mathcal{M} , a deterministic $(\text{SUP}; \text{SUM}^+)$ -automaton \mathbb{A} , and a threshold λ in binary $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in exponential time.*

We show that the distribution question for deterministic $(\text{SUP}; \text{SUM}^+)$ -automata is decidable in EXPTIME, i.e., (2) from Lemma 22. Decidability of the expected question for deterministic $(\text{SUP}; \text{SUM}^+)$ -automata is left as an open question.

Lemma 39. *The distribution question for deterministic $(\text{SUP}; \text{SUM}^+)$ -automata can be computed in exponential time.*

Proof. Let \mathbb{A} be a deterministic $(\text{SUP}; \text{SUM}^+)$ -automaton and let \mathcal{M} be a Markov chain. Let λ be a threshold in the distribution question. Consider \mathbb{A}' defined as \mathbb{A} considered as a $(\text{SUP}; \text{SUM}^B)$ -automaton with $B = \lambda + 1$. Observe that for every word w we have $\mathbb{A}(w) \leq \lambda$ iff $\mathbb{A}'(w) \leq \lambda$. Therefore, $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda) = \mathbb{D}_{\mathcal{M}, \mathbb{A}'}(\lambda)$. The latter value can be computed in exponential time (Lemma 18). \square

14.3 The upper bound for the approximation problems with $g = \text{Sum}$

Lemma 20. *Let $g \in \{\text{SUM}^+, \text{SUM}\}$. Given $\epsilon > 0$, a Markov chain \mathcal{M} , a deterministic $(\text{INF}; g)$ -automaton (resp., $(\text{SUP}; g)$ -automaton) \mathbb{A} , a threshold λ , both $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed up to precision ϵ in exponential time, and the dependency on ϵ is linear in the binary representation of ϵ .*

Key ideas. In end SCCs, deterministic $(\text{INF}; \text{SUM})$ -automata collect the same values as their variants with LIMINF as the master value function. Thus, the only difference in the values of a $(\text{INF}; \text{SUM})$ -automaton and $(\text{LIMINF}; \text{SUM})$ -automata comes from finite prefix of the word, before an end SCC of $\mathcal{M} \times \mathcal{A}_{mas}$ is reached, which happens with high probability after exponentially many steps N . We show that for exponential D , with high probability, all slave automata invoked in first N steps of the master automaton, terminate after at most D steps. Therefore, all values of these slave automata are bounded from below by $C \cdot D$, where C is the minimal weight in all slave automata. Thus, with high probability, a deterministic $(\text{INF}; \text{SUM})$ -automaton returns the value from a bounded interval. Thus, the sum value function can be replaced with the bounded sum and we can invoke Lemma 18.

Proof. Consider a deterministic $(\text{INF}; \text{SUM})$ -automaton \mathbb{A} . Let \mathbb{A}^{lim} be \mathbb{A} considered as a $(\text{LIMINF}; \text{SUM})$ -automaton. First, we assume that \mathbb{A}^{lim} has finite expected value. We can check whether this assumption holds in polynomial time by computing $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{lim})$ (Lemma 14). Then, we show the following **claim**: for every $\epsilon > 0$, there exists $B > 0$, exponential in $|\mathbb{A}| + |\log(\epsilon)|$ such that for \mathbb{A}' defined as \mathbb{A} considered as an $(\text{INF}; \text{SUM}^B)$ -automaton we have $|\mathbb{E}_{\mathcal{M}}(\mathbb{A}) - \mathbb{E}_{\mathcal{M}}(\mathbb{A}_B)| \leq \epsilon$.

The claim implies the lemma. Observe that due to Lemma 18 with the following remark on B , the expected value $\mathbb{E}_{\mathcal{M}}(\mathbb{A}_B)$ can be computed in polynomial time in \mathbb{A}_B , hence exponential time in \mathbb{A} . Therefore, we can approximate $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ up to ϵ in exponential time. Due to Markov inequality, for every λ we have $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda + \epsilon) - \mathbb{D}_{\mathcal{M}, \mathbb{A}_B}(\lambda - \epsilon) < \epsilon$. However, the values of \mathbb{A} are integers, therefore for $\epsilon < 0.5$ we get $|\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda) - \mathbb{D}_{\mathcal{M}, \mathbb{A}_B}(\lambda)| < \epsilon$. Therefore, again by Lemma 18, we can approximate $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ in exponential time in \mathbb{A} and polynomial in ϵ .

The proof of the claim. First, we observe that every run ends up in some end SCC of $\mathcal{A}_{mas} \times \mathcal{M}$, and, hence, Lemma 13 implies that values of all words are bounded from above by $|\mathbb{A}|$. Next, the values of all slave automata invoked in end strongly connected components (SCCs) of $\mathcal{A}_{mas} \times \mathcal{M}$ are bounded from below. Otherwise, the expected value of \mathbb{A} as a $(\text{LIMINF}; \text{SUM})$ -automaton is $-\infty$. Assume that the values of all slave automata invoked in end SCCs of $\mathcal{A}_{mas} \times \mathcal{M}$ are bounded from below, which implies that they are bounded by $-|\mathbb{A}|$. Then, we need to estimate the influence on the expected value of the slave automata invoked before the master automaton reaches an end SCC of $\mathcal{A}_{mas} \times \mathcal{M}$. Let E_1 be the expected value of a slave automaton of \mathbb{A} below $-B$, i.e., the expected value of the random variable $X_B(w) = \min(0, \mathfrak{B}(u) - B)$ for any slave automaton \mathfrak{B} over any subword u of w , and let E_2 be the expected number of steps of the master automaton before it reaches an end SCC. It follows that for $B > |\mathbb{A}|$, $|\mathbb{E}_{\mathcal{M}}(\mathbb{A}) - \mathbb{E}_{\mathcal{M}}(\mathbb{A}_B)| < |E_1| \cdot E_2$. To estimate $|E_1|$ we estimate the expected number of steps of a slave automaton exceeding B , i.e., the expected value of random variable $Y_B(u)$ defined as the maximum of 0 and the number of steps of a slave automaton minus B .

Let p be the minimal probability that occurs in \mathcal{M} and let $n = |\mathbb{A}|$. We show that for $B > \frac{n}{p^n} |\log \frac{n^2}{p^n} \epsilon|$, we have $\mathbb{E}(Y_B) \cdot E_2 < \epsilon$, which implies $|E_1| \cdot E_2 < \epsilon$. We show the estimate on E_2 first. Observe that starting from every state, there exists at least one word of length at most $|\mathcal{A}_{mas}|$ upon which the master automaton reaches an end SCC of $\mathcal{A}_{mas} \times \mathcal{M}$. Therefore, the master automaton reaches an end SCC in $|\mathcal{A}_{mas}|$ steps with probability at least $p^{|\mathcal{A}_{mas}|}$, and, hence, the number of steps before \mathcal{A}_{mas} reaches an end SCC is estimated from above by $|\mathcal{A}_{mas}|$ multiplied by the geometric distribution with the parameter $p^{|\mathcal{A}_{mas}|}$. Hence, E_2 is bounded by $\frac{n}{p^n}$. Now, we estimate $\mathbb{E}(Y_B)$. Observe that for every reachable state q of any slave automaton \mathfrak{B} , there exists a word of the length at most $|\mathfrak{B}|$ such that \mathfrak{B} , starting in q terminates

upon reading that word. Therefore, the probability $q_l(\mathfrak{B})$ that \mathfrak{B} works at least l steps is bounded by $(1 - p^{|\mathfrak{B}|})^{\lfloor \frac{l}{|\mathfrak{B}|} \rfloor}$. Now, $\mathbb{E}(Y_B)$ is bounded by the maximum over slave automata \mathfrak{B} of $\sum_{l \geq B} q_l(\mathfrak{B})$. We have $\sum_{l \geq B} q_l(\mathfrak{B}) \leq \frac{n}{p^n} \cdot (1 - p^n)^{\frac{B}{n}}$. Hence, $\mathbb{E}(Y_B) \leq \frac{n}{p^n} \cdot (1 - p^n)^{\frac{B}{n}}$ and $\mathbb{E}(Y_B) \cdot E_2 \leq \frac{n^2}{p^n} \cdot (1 - p^n)^{\frac{B}{n}}$. Observe that for $B > \frac{n}{p^n} s$, where $s = \lceil \log \frac{n^2}{p^n} \epsilon \rceil$, we have $\frac{n^2}{p^n} \cdot (1 - p^n)^{\frac{B}{n}} \leq \frac{n^2}{p^n} \cdot (\frac{1}{2})^s$ and $\mathbb{E}(Y_B) \cdot E_2 \leq \epsilon$. Observe that $\frac{n}{p^n} \cdot \lceil \log \frac{n^2}{p^n} \epsilon \rceil$ is exponential in $|\mathbb{A}| + \log(|\epsilon|)$ and linear in $|\log \epsilon|$.

Lifting the assumption. Now, we discuss how to remove the assumption that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{lim})$ is finite. For the expected question, observe that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) \leq \mathbb{E}_{\mathcal{M}}(\mathbb{A}^{lim})$, hence if the latter is $-\infty$, we can return the answer $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = -\infty$. For the distribution question, consider threshold λ . Observe that for every w , we have $\mathbb{A}(w) \leq \mathbb{A}_B(w)$. Moreover, $\mathbb{A}(w)$ is less than λ while $\mathbb{A}_B(w) \geq \lambda$ holds only if there is a slave automaton run before \mathcal{A}_{mas} reaches an end SCC which runs more than B steps. Therefore, the probability $\mathbb{P}_{\mathcal{M}}(\{w : \mathbb{A}(w) \leq \lambda \wedge \mathbb{A}_B(w) \geq \lambda\})$ is bounded from above by $\mathbb{E}(Y_B) \cdot E_2$. Thus, by the previous estimate on $\mathbb{E}(Y_B) \cdot E_2$, for $B > \max(\lambda + 1, \frac{n}{p^n} \log \frac{n^2}{p^n} \epsilon)$ we have $\mathbb{P}_{\mathcal{M}}(\{w : \mathbb{A}(w) \leq \lambda \wedge \mathbb{A}_B(w) \geq \lambda\}) < \epsilon$ and $|\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda) - \mathbb{D}_{\mathcal{M}, \mathbb{A}_B}(\lambda)| < \epsilon$. Again, $\mathbb{D}_{\mathcal{M}, \mathbb{A}_B}(\lambda)$ can be computed in exponential time in $|\mathbb{A}|$. \square

15 The proofs from Section 6.4

We begin with the remaining part of the proof of Lemma 26. Recall that the weighted Markov chain $\mathcal{M}^{\mathbb{A}}$ is defined as the product $\mathcal{A}_{mas} \times \mathcal{M}$, where \mathcal{A}_{mas} is the master automaton of \mathbb{A} and the weights of $\mathcal{M}^{\mathbb{A}}$ are the expected values of invoked slave automata. More precisely, the weight of the transition $\langle (q, s), a, (q', s') \rangle$ is the expected value of \mathfrak{B}_i , the slave automaton started by \mathcal{A}_{mas} in the state q upon reading a , w.r.t. the distribution given by \mathcal{M} starting in s .

Lemma 40. *Let \mathbb{A} be a deterministic (LIMAVG;SUM)-automaton. The values $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{E}(\mathcal{M}^{\mathbb{A}})$ coincide.*

In the following we prove Lemma 40. First, we show that lemma for (LIMAVG;SUM)-automata in which duration of runs of slave automata is bounded by N . Next, we show how to solve the general case of all (LIMAVG;SUM)-automata by the reduction to this special case. Before we continue, we discuss computing the expected values of Markov chains with silent moves.

Expected limit averages of Markov chains with silent moves. Let \mathcal{M}_{sil} be a Markov chain with weights from $\mathbb{N} \cup \{\lambda\}$, where λ corresponds to a silent transition. We consider the limit average value function with silent moves $\text{sil}(\text{LIMAVG})$, which applied to a sequence $a_1 a_2 \dots$ of elements of $\mathbb{N} \cup \{\lambda\}$ removes all λ symbols and applies the standard LIMAVG function to the sequence consisting of the remaining elements. The expected value of the limit average of a path in \mathcal{M}_{sil} can be computed by a slight modification of a standard method [24] for Markov chains without silent transitions. Namely, we associate with each transition (s, a, s') of \mathcal{M}_{sil} a real-valued variable $x[(s, a, s')]$. Next, we state the following equations:

- (1) for every transition (s, a, s') we put $x[(s, a, s')] = \sum_{s'' \in S_{\mathcal{M}_{sil}}, a' \in \Sigma} E((s', a', s''))$, and
- (2) $x[e_1] + \dots + x[e_k] = 1$, where e_1, \dots, e_k are all non-silent transitions, and the following inequalities
- (3) $0 \leq x[e] \leq 1$ for every transition e .

Then, following the argument for Markov chains without silent moves [24], we can show that the expected limit average of \mathcal{M}_{sil} is given as $c(e_1) \cdot x[e_1] + \dots + c(e_k) \cdot x[e_k]$ (once again e_1, \dots, e_k are all non-silent transitions).

15.1 The expected value in the bounded-duration case

First, we show that Lemma 40 holds if we assume that for some $N > 0$ all slave automata take at most N transitions.

Lemma 41. *Let \mathbb{A} be a (LIMAVG;SUM)-automaton in which duration of runs of slave automata is bounded by N and let $\mathcal{M}^{\mathbb{A}}$ be the Markov chain corresponding to \mathbb{A} . The values $\mathbb{E}_{\mathcal{M}}(\mathbb{A})$ and $\mathbb{E}(\mathcal{M}^{\mathbb{A}})$ coincide.*

The plan of the proof. We define a $\text{sil}(\text{LIMAVG})$ -automaton \mathcal{A} that simulates runs of \mathbb{A} ; the value on \mathcal{A} on every word coincides with \mathbb{A} . Then, we transform the Markov chain $\mathcal{A} \times \mathcal{M}$ into a Markov chain

\mathcal{M}_E by adjusting its weights only. We change all weights to silent weight λ except for the transitions corresponding to the invocation of slave automata, where the weight is the expected value of the invoked slave automaton w.r.t. the distribution given by \mathcal{M} in the current state. In the proof we argue that the expected values of limit average of $\mathcal{A} \times \mathcal{M}$ and \mathcal{M}_E coincide. We show that by looking at the linear equations corresponding to computing the expected limit average of each of the Markov chains. Basically, the frequency of each transition is the same in both Markov chains and changing the value of the slave automaton from its actual value to the expected value does not affect the solution to the set of equations. Next, we observe that runs of slave automata past the first transition do not matter. Indeed, all runs of slave automata are accepting and all weights past the first transition are 0. Thus, we can reduce \mathcal{M}_E to \mathcal{M}_R by projecting out information about the runs of slave automata past the first transition. Finally, we observe that such a Markov chain \mathcal{M}_R is in fact $\mathcal{M}^{\mathbb{A}}$. Hence, we have shown that

$$\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}_{\mathcal{M}}(\mathcal{A}) = \mathbb{E}(\mathcal{M}_E) = \mathbb{E}(\mathcal{M}_R) = \mathbb{E}(\mathcal{M}^{\mathbb{A}})$$

Proof. Every slave automaton of \mathbb{A} takes at most N steps. Therefore, \mathbb{A} has width bounded by N . Moreover, without loss of generality, we assume that each slave automaton takes transitions of weight 0 except for the last transition, which may have a non-zero weight, and all slave automata are either trivial, i.e., they start in the accepting state and take no transitions, or they take precisely N transitions. Basically, slave automata may keep track of the accumulated values and the number of steps in their states.

The automaton \mathcal{A} . Let Q_{mas} be the set of states of the master automaton of \mathbb{A} and let Q_s be the union of the set of states of the slave automata of \mathbb{A} . We define \mathcal{A} as a $\text{sil}(\text{LIMAVG})$ automaton over the set of states $Q_{mas} \times (Q_s \cup \{\perp\})^N$. The component Q_{mas} is used to keep track of the run of the master automaton while the component $(Q_s \cup \{\perp\})^N$ is used to keep track of up to N slave automata running concurrently. The symbol \perp corresponds to an empty slot that can be used to simulate another slave automaton. Since \mathbb{A} has width bounded by N , the automaton \mathcal{A} can simulate the Boolean part of the run of \mathbb{A} . The weight of a transition of \mathcal{A} is either λ if not automaton terminates or it is the value of terminating slave automaton (non-trivial slave automata take precisely N steps, so at most one can terminate at each position). Transitions at which no slave automaton terminates are silent transitions. The automata \mathbb{A} and \mathcal{A} encounter the same weights but differ in their aggregation. The value of a slave automaton is associated to the position at which it is invoked, while in \mathcal{A} it is associated with the position at which the slave automaton terminates. However, these positions differ by N , therefore the limit average of both sequences coincides. Hence, for every word w , the values $\mathbb{A}(w)$ and $\mathcal{A}(w)$ coincide. It follows that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}_{\mathcal{M}}(\mathcal{A})$.

The Markov chain \mathcal{M}_E . We define \mathcal{M}_E as $\mathcal{A} \times \mathcal{M}$ with altered weights defined as follows. All transitions which correspond to the invocation of a slave automaton \mathfrak{B}_i with the state of the Markov chain \mathcal{M} being s have weight equal to the expected value of \mathfrak{B}_i w.r.t. the distribution given by \mathcal{M} starting in the state s . Other transitions are silent.

Expected value of $\mathcal{A}_{mas} \times \mathcal{M}$ and \mathcal{M}_E coincide. Recall that the expected limit average of a Markov chain with silent moves is given by $c(e_1) \cdot x[e_1] + \dots + c(e_k) \cdot x[e_k]$ where variables $x[e]$, over all transitions e , form a solution to the system of equations and inequalities (1), (2) and (3), and e_1, \dots, e_k are all non-silent transitions. Now, observe that the equations (1) and inequalities (2) are the same for both Markov chains $\mathcal{A} \times \mathcal{M}$ and \mathcal{M}_E . The equation (2) is, in general, different for $\mathcal{A} \times \mathcal{M}$ and for \mathcal{M}_E . However, non-silent transitions of $\mathcal{A} \times \mathcal{M}$, denoted by e_1, \dots, e_k , are all states at which at least one slave automaton terminates, while non-silent transitions of \mathcal{M}_E , denoted by e'_1, \dots, e'_l are all states at which some (non-trivial) slave automaton is invoked. Observe that every terminating slave automaton has been invoked, and, in \mathcal{A} , every invoked slave automaton terminates. Therefore, the cumulated frequency of invocations and terminations of slave automata coincides, i.e., equations (1) imply $x[e_1] + \dots + x[e_k] = x[e'_1] + \dots + x[e'_l]$. It follows that equations (1), (2) and (3) corresponding to $\mathcal{A} \times \mathcal{M}$ and to \mathcal{M}_E have the same solution. It remains to show that $c(e_1) \cdot x[e_1] + \dots + c(e_k) \cdot x[e_k] = c'(e'_1) \cdot x[e'_1] + \dots + c'(e'_l) \cdot x[e'_l]$, where c (resp. c') are weights in $\mathcal{A} \times \mathcal{M}$ (resp., \mathcal{M}_E).

Since $c'(e')$ is the expected value of the slave automaton started at e' , the expected value $c'(e')$ is given by $c'(e') = \sum_{e'' \in T} p(e', e'') c(e'')$, where T is the set of transitions that correspond to the final transitions of the slave automaton started at the transition e' , and $p(e', e'')$ is the probability of reaching the transition e'' from e' omitting the set T . Indeed, each (non-trivial) slave automaton takes precisely N transition, hence at each position at most one non-trivial slave automaton terminates and $c(e'')$ is the

value of the slave automaton terminating at e'' . Therefore, $c'(e') = \sum_{e'' \in T} p(e', e'')c(e'')$. Now, we take $c'(e'_1) \cdot x[e'_1] + \dots + c'(e'_l) \cdot x[e'_l]$ and substitute each $c(e'_i)$ by the corresponding $c'(e'_i) = \sum_{e'' \in T_i} p(e', e'')c(e'')$. Then, we now group in all the terms by e'' , i.e., we write it as $c(e_1)(x[e'_1]p(e'_1, e_1) + \dots + x[e'_l]p(e'_l, e_1)) + \dots$. Observe that the frequency of taking the transition e_1 at which some slave automaton \mathfrak{B} terminates is equal to the sum of frequencies on transitions at which this slave automaton \mathfrak{B} has been invoked weighted by the probability of reaching the terminating transition e_1 from a given invoking transition. Therefore, we have $x[e'_1]p(e'_1, e_1) + \dots + x[e'_l]p(e'_l, e_1) = x[e_1]$. It follows that $c(e_1) \cdot x[e_1] + \dots + c(e_k) \cdot x[e_k] = c'(e'_1) \cdot x[e'_1] + \dots + c'(e'_l) \cdot x[e'_l]$ and $\mathbb{E}_{\mathcal{M}}(\mathcal{A}) = \mathbb{E}(\mathcal{M}_E)$.

The Markov chain \mathcal{M}_R . We construct \mathcal{M}_R from \mathcal{M}_E by projecting out the component $(Q_s \cup \{\perp\})^N$. We claim that this step preserves the expected value. First, observe that the distribution is given by an unaffected component \mathcal{M} and the weights depend only on the state of the Markov chain \mathcal{M} and the state of the master automaton \mathcal{A}_{mas} . Thus, projecting out the component $(Q_s \cup \{\perp\})^N$ does not affect the expected value, i.e., $\mathbb{E}_{\mathcal{M}}(\mathcal{M}_E) = \mathbb{E}(\mathcal{M}_R)$. Now, observe that the set of states of \mathcal{M}_R is $Q_{mas} \times Q_{\mathcal{M}}$. Observe that the probability and the weights of the transitions of \mathcal{M}_R match the conditions of the definition of $\mathcal{M}^{\mathbb{A}}$. Therefore, $\mathcal{M}_R = \mathcal{M}^{\mathbb{A}}$. \square

15.2 Reduction to the bounded-duration case

Let \mathbb{A} be a (LIMAVG;SUM)-automaton. For every N , we define \mathbb{A}_N as \mathbb{A} with the bound N imposed on slaves, i.e., each slave automaton terminates either by reaching an accepting state or when it takes N -th step. Let $\mathcal{M}_N^{\mathbb{A}}$ be the Markov chain that corresponds to \mathbb{A}_N . Observe that as N tends to infinity, weights in $\mathcal{M}_N^{\mathbb{A}}$ converge to the weights in $\mathcal{M}^{\mathbb{A}}$. It remains to be shown that, as N tends to infinity, the expected values of \mathbb{A}_N converge to the expected value of \mathbb{A} . We show in the following Lemma 42 that random variables generated by \mathbb{A}_N converge in probability to the random variable generated by \mathbb{A} , i.e., for every $\epsilon > 0$ we have

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mathcal{M}}(\{w : |\mathbb{A}(w) - \mathbb{A}_N(w)| \geq \epsilon\}) = 0$$

Convergence in probability implies convergence of the expected values. It follows that the expected values of \mathbb{A} and $\mathcal{M}^{\mathbb{A}}$ coincide.

Lemma 42. *The random variables defined by $\{\mathbb{A}_N\}_{N \geq 1}$ converge in probability to the random variable defined by \mathbb{A} .*

Proof. We define an (LIMAVGSUP;SUM)-automaton $\mathbb{A}^{\geq N}$ as the automaton obtained from \mathbb{A} in the following way. First, each slave automaton take transitions of weight 0 for the first (up to) N steps, past which it takes transitions of weight 1 until it terminates. Second, the value function of the master automaton is LIMAVGSUP defined on a_1, a_2, \dots as $\text{LIMAVGSUP}(a_1 \dots) = \limsup_n \frac{1}{n} \sum_{i=1}^n a_i$. Intuitively, the automaton $\mathbb{A}^{\geq N}$ computes the limit average (supremum) of the steps slave automata take above the threshold N . Let C be the maximal absolute weight in slave automata of \mathbb{A} . Then, for every word w we have

$$\mathbb{A}_N(w) - C \cdot \mathbb{A}^{\geq N}(w) \leq \mathbb{A}(w) \leq \mathbb{A}_N(w) + C \cdot \mathbb{A}^{\geq N}(w).$$

It follows that

$$\mathbb{P}_{\mathcal{M}}(\{w : |\mathbb{A}(w) - \mathbb{A}_N(w)| \geq \epsilon\}) = \mathbb{P}_{\mathcal{M}}(\{w : |\mathbb{A}^{\geq N}(w)| \geq \frac{\epsilon}{C}\})$$

We show that with N increasing to infinity, $\mathbb{P}_{\mathcal{M}}(\{w : |\mathbb{A}^{\geq N}(w)| \geq \frac{\epsilon}{C}\})$ converge to 0. From that we conclude that \mathbb{A}_N converge in probability to \mathbb{A} as N tends to infinity.

Observe that for every word w and every N we have $0 \leq \mathbb{A}^{\geq N}(w)$ and $\mathbb{A}^{\geq N}(w) \geq \mathbb{A}^{\geq N+1}(w)$. Therefore, we only need to show that for every $\epsilon > 0$ there for N large enough $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{\geq N}) \leq \epsilon$. Then, by Markov inequality, $\mathbb{P}_{\mathcal{M}}(\{w : |\mathbb{A}^{\geq N}(w)| \geq \sqrt{\epsilon}\}) < \sqrt{\epsilon}$.

To estimate the value of $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{\geq N})$ we consider $\text{sil}(\text{LIMAVGSUP})$ -automata $\mathbb{A}[K, i]$ defined as follows. The automaton $\mathbb{A}[K, i]$ simulates the master automaton \mathbb{A} and slaves that are invoked at positions $\{K \cdot l + i : l \in \mathbb{N}\}$. For every $l > 0$, the transition at the position $K \cdot (l + 1) + i$ has the weight 1 if the slave invoked at the position $K \cdot l + i$ works for at least K steps. Otherwise, this transition has weight 0. On the remaining positions, transitions have weight 0. Observe that due to distributivity of the limit supremum, the limit average supremum of the number of slave automata that take at least

K steps at a given word w is bounded by $\sum_{i=0}^{K-1} \mathbb{A}[K, i]$. It follows that for every word w we have $\mathbb{A}^{\geq N}(w) \leq \sum_{K \geq N} \sum_{i=0}^{K-1} \mathbb{A}[K, i](w)$. Therefore,

$$(*) \mathbb{E}_{\mathcal{M}}(\mathbb{A}^{\geq N}) \leq \sum_{K \geq N} \sum_{i=0}^{K-1} \mathbb{E}_{\mathcal{M}}(\mathbb{A}[K, i]).$$

Now, we estimate $\mathbb{E}_{\mathcal{M}}(\mathbb{A}[K, i])$. Let n be the maximal size of a slave automaton in \mathbb{A} and let k be the number of slave automata. We assume, without loss of generality, that every state of slave automata is reached along some run on words generated by \mathcal{M} . Now, observe that from every state of slave automata some accepting state is reachable. Otherwise, there would be a set of strictly positive probability at which \mathbb{A} does not accept. Moreover, as it is reachable, it is reachable within n steps. Therefore, there exists a probability $p > 0$ such that any slave automaton in any state terminates after next n steps. It follows that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}[K, i]) \leq \frac{1}{K} p^{\lfloor \frac{K}{n} \rfloor}$. With that estimate, we obtain from $(*)$ that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{\geq N}) \leq \sum_{K \geq N} p^{\lfloor \frac{K}{n} \rfloor} \leq n \cdot \frac{p^{\lfloor \frac{N}{n} \rfloor}}{1-p}$. Therefore, $\mathbb{E}_{\mathcal{M}}(\mathbb{A}^{\geq N})$ converges to 0 as N increases to infinity. Finally, this implies that \mathbb{A}_N converge in probability to \mathbb{A} as N tends to infinity. \square

15.3 The distribution question

Lemma 27. *Let $g \in \text{FinVal}$. Given a Markov chain \mathcal{M} , a deterministic $(\text{LIMAVG}; g)$ -automaton \mathbb{A} and a value λ , the value $\mathbb{D}_{\mathcal{M}, \mathbb{A}}(\lambda)$ can be computed in polynomial time.*

Proof. Let \mathbb{A} be a deterministic $(\text{LIMAVG}; \text{SUM})$ -automaton with the master automaton \mathcal{A}_{mas} and let \mathcal{M} be a Markov chain. Moreover, let $\mathcal{M}^{\mathbb{A}}$ be the Markov chain obtained from \mathcal{M} and \mathbb{A} . We show that the distribution $\mathbb{D}_{\mathcal{M}, \mathbb{A}}$ and the distribution defined by $\mathcal{M}^{\mathbb{A}}$ coincide.

A single SCC case. Assume that $\mathcal{M} \times \mathcal{A}_{mas}$ is an SCC. Observe that event “the value of \mathbb{A} equals λ ” is a tail event w.r.t. the Markov chain \mathcal{M} , i.e., it does not depend on finite prefixes. Therefore, its probability is either 0 or 1 [23]. It follows that the value of almost all words is equal to the expected value of \mathbb{A} . Now, $\mathcal{M}^{\mathbb{A}}$ is structurally the same as $\mathcal{M} \times \mathcal{A}_{mas}$, hence it is also an SCC. Therefore, also in $\mathcal{M}^{\mathbb{A}}$ almost all words have the same value, which is equal to $\mathbb{E}(\mathcal{M}^{\mathbb{A}})$. As $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}(\mathcal{M}^{\mathbb{A}})$ (Lemma 40) we have $\mathbb{D}_{\mathcal{M}, \mathbb{A}}$ and the distribution defined by $\mathcal{M}^{\mathbb{A}}$ coincide.

The general case. Consider the case where $\mathcal{M} \times \mathcal{A}_{mas}$ consists of multiple end SCC S_1, \dots, S_k . Using conditional probability, we can repeat the single-SCC-case argument to show that in each end SCC S_1, \dots, S_k the values of \mathbb{A} are the same and equal to the expected values in these SCC. Similarly, in each end SCC of $\mathcal{M}^{\mathbb{A}}$, all words have the same value, which is equal to the expected value in that SCC. Since $\mathcal{M} \times \mathcal{A}_{mas}$ is structurally the same as $\mathcal{M}^{\mathbb{A}}$, each SCC S_1, \dots, S_k corresponds to an SCC in $\mathcal{M}^{\mathbb{A}}$. Lemma 40 states that $\mathbb{E}_{\mathcal{M}}(\mathbb{A}) = \mathbb{E}(\mathcal{M}^{\mathbb{A}})$. By applying Lemma 40 to \mathcal{M} and \mathbb{A} with different initial states of \mathcal{M} and \mathcal{A}_{mas} (in each S_1, \dots, S_k), we infer that in every SCC S_1, \dots, S_k the expected values of \mathbb{A} and $\mathcal{M}^{\mathbb{A}}$ coincide. Therefore, the distribution $\mathbb{D}_{\mathcal{M}, \mathbb{A}}$ and the distribution defined by $\mathcal{M}^{\mathbb{A}}$ coincide. \square

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